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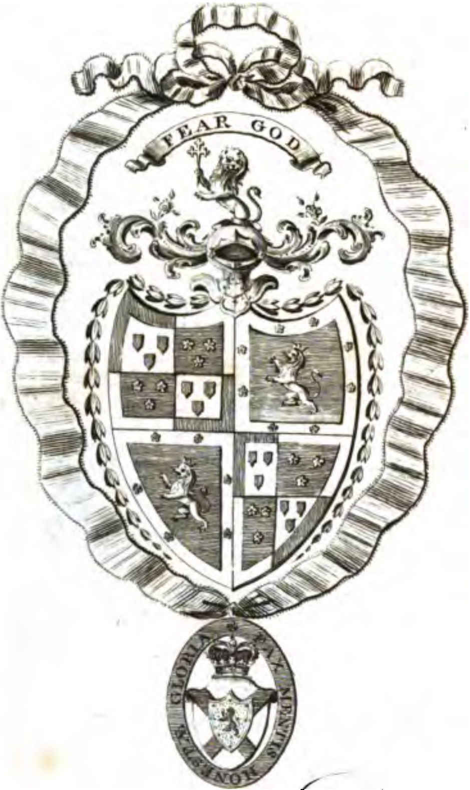

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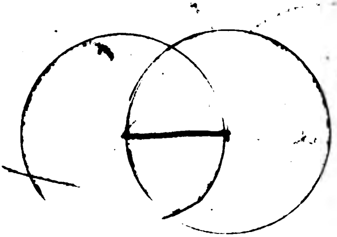

D3

EUCLIDE's
ELEMENTS;
The whole Fifteen BOOKS
Compendiouſly Demonſtrated.

WITH

ARCHIMEDES *Theorems*
Of the Sphere and Cylinder, inveſtigated by the *Method of Indiviſibles.*

By ISAAC BARROW, *D. D. Late Maſter of* Trinity *College in* Cambridge.

To which is added in this Edition,

EUCLIDE's DATA
with *Marinus*'s Preface.
And a Brief TREATISE of
REGULAR SOLIDS.

Καθαρμοὶ ψυχῆς λογικῆς εἰσὶν αἱ μαθηματικαὶ ἐπιστῆμαι.

LONDON: Printed and Sold by *W. Redmayne*
in *Jewen-ſtreet*, *R. Mount* on *Tower-hill*, and *J.* and
B. Sprint in *Little-britain.* 1714.

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To the READER.

IF you are deſirous, Courteous Reader, to know what I have perform'd in this Edition of the Elements of Euclide, I ſhall here explain it to you in ſhort, according to the nature of the Work. I have endeavour'd to attain two ends chiefly; the firſt, to be very perſpicuous, and at the ſame time ſo very brief, that the Book may not ſwell to ſuch a Bulk, as may be troubleſome to carry about one, in which I think I have ſucceeded, unleſs in my abſence the Printer's *care ſhould fruſtrate my Deſign. Some of a brighter* Genius, *and endued with greater Skill, may have demonſtrated moſt of theſe Propoſitions with more nicety, but perhaps none with more ſuccinctneſs than I have; eſpecially ſince I alter'd nothing in the number and order of the Author's Propoſitions; nor preſum'd either to take the liberty of rejecting, as leſs neceſſary, any of them, or of reducing ſome of the eaſier ſort into the rank of Axioms, as ſeveral have done; and among others, that moſt expert Geometrician* A. Tacquetus C. *(whom I the more willingly name, becauſe I think it is but civil to acknowledge that I have imitated him in ſome Points) after whoſe moſt acurate Edition I had no Thoughts of attempting any thing of this nature, till I conſider'd that this moſt learned Man thought fit to publiſh only eight of* Euclide's *Books, which he took the pains to explain and embelliſh, having in a manner rejected and undervalued the other ſeven, as leſs appertaining to the Elements of Geometry. But my Province was* *originally quite different, not that of writing the Elements of Geometry, after what method ſoever I pleas'd, but of demonſtrating, in as few words as poſſible I cou'd, the whole Works of* Euclide. *As to four of the Books, viz. the ſeventh, eighth, ninth, and tenth, altho' they don't ſo nearly appertain to the Elements of plain and ſolid Geometry, as the ſix precedent and the two ſubſequent, yet none of the more skilful Geometricians can be ſo ignorant as not to know that they are very uſeful for Geometrical matters, not only by reaſon of the mighty near affinity that is between Arithmetick and Geometry, but alſo for the knowledge of both meaſurable and unmeaſurable Magnitudes, ſo exceeding neceſſary for the Doctrine of both plain and ſolid figures. Now the noble Contemplation of the five Regular Bodies that is contain'd in the three laſt Books, cannot without great Injuſtice be pretermitted, ſince that for the ſake thereof our* ϛοιχειωτὴς, *being a Philoſopher of the Platonic Sect, is ſaid to have compos'd this univerſal Syſtem of Elements; as* Proclus lib. 2. *witneſſeth in theſe Words,* Ὅθεν δὴ ϗ τῆς συμπάσης ϛοιχειώσεως τέλος προεϛήσατο τὴν τῶν καλȣμένων πλατωνικῶν χημάτων σύϛασιν. *Beſides, I eaſily perſwaded my ſelf to think, that it would not be unacceptable to any Lover of theſe Sciences to have in his Poſſeſſion the whole Euclidean Work, as it is commonly cited and celebrated by all Men. Wherefore I reſolv'd to omit no Book or Propoſition of thoſe that are found in* P. Herigonius's *Edition, whoſe Steps I was oblig'd cloſely to follow, by reaſon I took a Reſolution to make uſe of moſt of the Schemes of the ſaid Book, very well foreſeeing that time would not allow me to form* *new ones, tho' ſometimes I choſe rather to do it. For the ſame reaſon I was willing to uſe for the moſt part* Euclide*'s own Demonſtrations, having only expreſs'd them in a more ſuccinct Form, unleſs perhaps in the ſecond, thirteenth, and very few in the ſeventh, eighth, and ninth Book, in which it ſeem'd not worth my while to deviate in any particular from him. Therefore I am not without good hopes that, as to this part I have in ſome meaſure ſatisfied both my own Intentions, and the Deſire of the Studious. As for ſome certain Problems and Theorems that are added in the Scholiums (or ſhort Expoſitions) either appertaining (by reaſon of their frequent Uſe) to the nature of theſe Elements, or conducing to the ready Demonſtration of thoſe things that follow, or which do intimate the reaſons of ſome principal Rules of practical Geometry, reducing them to their original Fountains, theſe I ſay, will not, I hope, make the Book ſwell to a Size beyond the deſign'd Proportion.*

The other Butt, which I levell'd at, is to content the Deſires of thoſe who are delighted more with ſymbolical than verbal Demonſtrations. In which kind, whereas moſt among us are accuſtom'd to the Symbols of Gulielmus Oughtredus, *I therefore thought beſt to make uſe, for the moſt part, of his. None hitherto (as I know of) has attempted to interpret and publiſh* Euclide *after this manner, except* P. Herigonius; *whoſe Method (tho' indeed moſt excellent in many things, and very well accommodated for the particular purpoſe of that moſt ingenious Man) yet ſeems in my Opinion to labour under a double Defect. Firſt, in regard that, altho' of two or more Propoſitions, produ-*

ced for the Proof of any one Problem or Theorem, the former don't always depend of the latter, yet it don't readily enough appear either from the order of each, or by any other manner, when they agree together, and when not; wherefore for want of the Conjunctions and Adjectives, ergo, rurſus, *&c. many difficulties and occaſions of doubt do often ariſe in reading, eſpecially to thoſe that are Novices. Beſides it frequently happens, that the ſaid Method cannot avoid ſuperfluous Repetitions, by which the Demonſtrations are oftentimes render'd tedious, and ſometimes alſo more intricate; which Faults my Method doth eaſily remedy by the arbitrary mixture of both Words and Signs. Therefore let what has been ſaid, touching the Intention and Method of this little Work, ſuffice. As to the reſt, whoever covets to pleaſe himſelf with what may be ſaid, either in Praiſe of the Mathematicks in general, or of Geometry in particular, or touching the Hiſtory of theſe Sciences, and conſequently of* Euclide *himſelf, (who digeſted thoſe Elements) and others* πάρεργα *of that kind, may conſult other Interpreters: Neither will I (as if I were afraid leaſt theſe my Endeavors may fall ſhort of being ſatisfactory to all Perſons) alledge as an Excuſe (tho' I may very lawfully do it) the want of due time which ought to be employ'd in this Work, nor the Interruption occaſion'd by other Affairs, nor yet the want of requiſite help for theſe Studies nor ſeveral other things of the like nature. But what I have here employ'd my Labour and Study in for the Uſe of the ingenuous Reader, I wholly ſubmit to his Cenſure and Judgment, to approve if uſeful, or reject if otherwiſe.*

J. B.

Ad amiciſſimum Virum, I. C. de EUCLIDE contracto, Εὐφημισμός.

FActum bene! didicit Laconice loqui
Senex profundus, & aphoriſmos induit.
Immenſa dudum margo commentarii
Diagramma circuit minutum; utque Inſula
Problema breve natabat in vaſto mari.
Sed unda jam detumuit; & gloſſa arctior
Stringit Theoremata; minoris anguli
Lateribus ecce totus Euclides *jacet,*
Incluſus olim velut Homerus in nuce;
Pluteoque ſarcina modo qui incubuit, levis
En fit manipulus. Pelle in exigua latet
Ingens Matheſis, matris utero Hercules,
In glande quercus, vel Ithaca Eurus in pila.
Nec mole dum decreſcit, uſu fit minor;
Quin auctior jam evadit, & cumulatius
Contracta prodeſt erudita pagina.
Sic ubere magis liquor è preſſo affluit;
Sic pleniori vaſa inundat ſanguinis
Torrente cordis Syſtole; ſic fuſius
Procurrit æquor ex Abylæ anguſtiis.
Tantilli operis ars tanta referenda unice eſt
BAROVIANO *nomini, ac ſolertiæ.*
Sublimis euge mentis ingenium potens!
Cui invitum nil, arduum eſſe nil ſolet;
Sic uſque pergas proſpero conamine,
Radiuſque multum debeat ac abacus tibi;
Sic creſcat indies feracior ſeges,
Simili colonum germine aſſiduo beans.
Specimen futuræ meſſis hic fiet labor.
Magnaque famæ illuſtria hæc præludia.
Juvenis dedit qui tanta, quid dabit ſenex?

Car. Robotham, *CANTAB.*
Coll. Trin. Sen. Soc.

The

The Explication of the Signs or Characters.

=	Signifies	Equal.
⊏		Greater.
⊐		Leſſer.
+		More, or to be added.
−		Leſs, or to be ſubtracted.
∼		The Differences, or Exceſs; Alſo, that all the quantities which follow, are to be ſubtracted, the Signs not being changed.
×		Multiplication, or the Drawing one ſide of a Rectangle into another. The ſame is denoted by the Conjunction of letters; as $AB = A \times B$.
√		The Side or Root of a Square, or Cube, &c.
Q & q		A Square.
C & c		A Cube.
Q.Q.		The ratio of a ſquare number to a ſquare number.

Other Abbreviations of words, where-ever they occur, the Reader will without trouble underſtand of himſelf; ſaving ſome few, which, being of leſs general uſe, we refer to be explained in their places.

The

THE FIRST BOOK OF EUCLIDE'S ELEMENTS.

Definitions.

I. A Point is that which has no part.

II. A Line is a longitude without latitude.

III. The ends, or limits, of a line are Points.

IV. A Right Line is that which lies equally betwixt its Points.

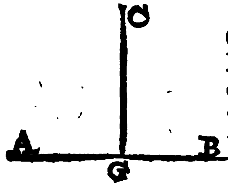
V. A Superficies is that which has only longitude and latitude.

VI. The extremes, or limits, of a Superficies are lines.

VII. A plain Superficies is that which lies equally betwixt its lines.

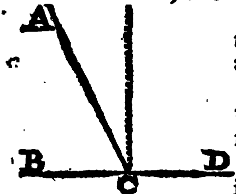
VIII. A plain Angle is the inclination of two lines the one to the other, the one touching the other in the ſame plain, yet not lying in the ſame ſtrait line.

IX. And if the lines which contain the Angle be right lines, it is called a right-lined Angle.



X. When a right line CG ſtanding upon a right line AB, makes the angles on either ſide thereof, CGA, CGB, equal one to the other, then both thoſe equal angles are right angles; and the right line CG, which ſtandeth on the other, is termed a Perpendicular to that (AB) whereon it ſtandeth.

Note, *When ſeveral angles meet at the ſame point (as at G) each particular angle is deſcribed by three letters; whereof the middle letter ſheweth the angular point, and the two other letters the lines that make that angle: As the angle which the right lines CG, AG make at G, is called CGA, or AGC.*



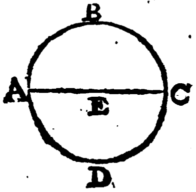
XI. An obtuſe angle is that which is greater than a right angle; as ACD.

XII. An acute angle is that which is leſs than a right angle; as ACB.

XIII. A Limit, or Term, is the end of any thing.

XIV. A Figure is that which is contained under one or more terms.

XV. A Circle is a plain figure contained under one line, which is called a Circumference; unto which all lines drawn from one point within the figure, and falling upon the circumference thereof, are equal the one to the other.



XVI. And that point is called the Center of the Circle.

XVII. A Diameter of a circle is a right line drawn through the center thereof, and ending at the circumference on either ſide, dividing the circle into two equal parts.

XVIII. A Semicircle is a figure which is contained under the diameter, and under that part of the circumference which is cut off by the diameter.

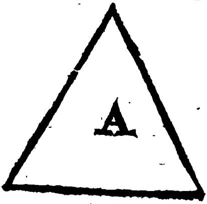
In the circle EABCD, E is the center, AC the diameter, ABC the ſemicircle.

XIX. Right-lined figures are ſuch as are contained under right lines.

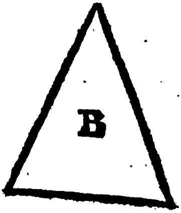
XX. Three-ſided or Trilateral figures are ſuch as are contained under three right lines.

XXI. Four-ſided or Quadrilateral figures are ſuch as are contained under four right lines.

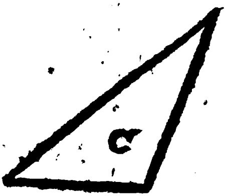
XXII. Many-ſided figures are ſuch as are contained under more right lines than four.



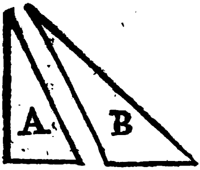
XXIII. Of Trilateral figures, that is an Equilateral Triangle, which hath three equal ſides; as the Triangle A.



XXIV. Iſoſceles is a Triangle which hath only two ſides equal; as the Triangle B.

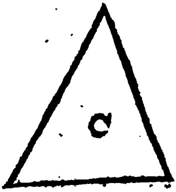


XXV. Scalenum is a Triangle whoſe three ſides are all unequal; as C.



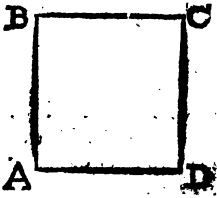
XXVI. Of theſe Trilateral figures, a right-angled Triangle is that which has one right angle; as the Triangle A.

XXVII. An Amblygonium, or obtuſe-angled Triangle, is that which has one angle obtuſe; as B.

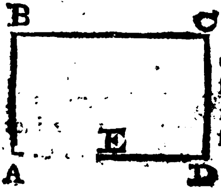


XXVIII. An Oxygonium, or acute-angled Triangle, is that which has three acute angles; as C.

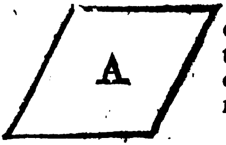
An Equiangular, or equal-angled figure is that whereof all the angles are equal. Two figures are equiangular, if the ſeveral angles of the one figure be equal to the ſeveral angles of the other. The ſame is to be underſtood of Equilateral figures.



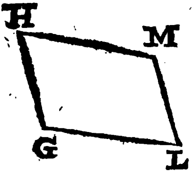
XXIX. Of Quadrilateral, or four-ſided figures, a Square is that whoſe ſides are equal, and angles right; as ABCD.



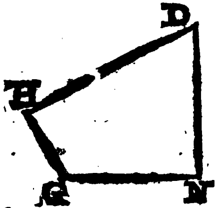
XXX. A figure on the one part longer, or a long ſquare, is that which hath right angles, but not equal ſides; as ABCD.



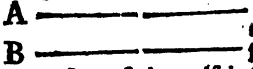
XXXI. A Rhombus, or diamond-figure, is that which has four equal ſides, but is not right-angled; as A.



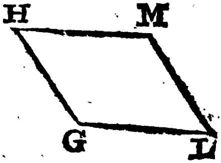
XXXII. A Rhomboides, or diamond-like figure, is that whoſe oppoſite ſides, and oppoſite angles, are equal; but has neither equal nor right angles; as GLMH.



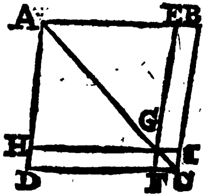
XXXIII. All other quadrilateral figures beſides theſe are called *Trapezia*, or Tables; as GNDH.



XXXIV. Parallel, or equidiſtant right lines are ſuch, which being in the ſame ſuperficies, if infinitely produced, would never meet; as A and B.



XXXV. A Parallelogram is a quadrilateral figure, whoſe oppoſite ſides are parallel, or equidiſtant; as GLHM.



XXXVI. In a Parallelogram ABCD, when a diameter AC, and two lines EF, HI parallel to the ſides, cutting the diameter in one and the ſame point G, are drawn, ſo that the Parallelogram be divided by them into four parallelograms; thoſe two DG, GB, through which the diameter paſſeth not, are called Complements; and the other two HE, FI, through which the diameter paſſeth, the Parallelograms ſtanding about the diameter.

A Problem *is, when ſomething is propoſed to be done or effected.*

A Theoreme *is, when ſomething is propoſed to be demonſtrated.*

A Corollary *is a conſectary, or ſome conſequent truth gained from a preceding demonſtration.*

A Lemma *is the demonſtration of ſome premiſe, whereby the proof of the thing in hand becomes the ſhorter.*

Poſtulates or Petitions.

1. FRom any point to any point to draw a right line.

2. To produce a right line finite, ſtrait forth continually.

3. Upon any center, and at any diſtance, to deſcribe a circle.

Axioms.

1. THings equal to the ſame third, are alſo equal one to the other.

As $A=B=C$. Therefore $A=C$. Or therefore all, A, B, C, are equal the one to the other.

Note, *When ſeveral quantities are joined the one to the other continually with this mark $=$, the firſt quantity is by virtue of this axiome equal to the laſt, and every one to every one: In which caſe we often ab-*

ſtain from citing the axiome, for brevity's ſake; altho' the force of the conſequence depend thereon.

2. If to equal things you add equal things, the wholes ſhall be equal.

3. If from equal things you take away equal things, the things remaining will be equal.

4. If to unequal things you add equal things, the wholes will be unequal.

5. If from unequal things you take away equal things, the remainders will be unequal.

6. Things which are double to the ſame third, or to equal things, are equal one to the other. Underſtand the ſame of triple, quadruple, &c.

7. Things which are half of one and the ſame thing, or of things equal, are equal the one to the other. Conceive the ſame of ſubtriple, ſubquadruple, &c.

8. Things which agree together, are equal one to the other.

The converſe of this axiome is true in right lines and angles, but not in figures; unleſs they be like.

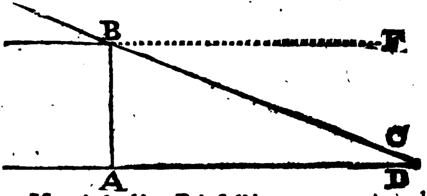
Moreover, magnitudes are ſaid to agree, when the parts of the one being applyed to the parts of the other, they fill up an equal or the ſame place.

9. Every whole is greater than its part.

10. Two right lines cannot have one and the ſame ſegment (or part) common to them both.

11. Two right lines meeting in the ſame point, if they be both produced, they ſhall neceſſarily cut one the other in that point.

12. All right angles are equal the one to the other.



13. If a right line BA falling on two right lines

AD, CB, make the internal angles on the ſame ſide, BAD, ABC, leſs than two right angles, thoſe two right lines produced ſhall meet on that ſide, where the angles are leſs than two right angles.

14. Two right lines do not contain a ſpace.

15. If to equal things, you add things unequal, the exceſs of the wholes ſhall be equal to the exceſs of the additions.

16. If to unequal things equal be added, the exceſs of the wholes ſhall be equal to the exceſs of thoſe which were at firſt.

17. If from equal things, unequal things be taken away, the exceſs of the remainders ſhall be equal to the exceſs of the wholes.

18. If from things unequal, things equal be taken away, the exceſs of the remainders ſhall be equal to the exceſs of the wholes.

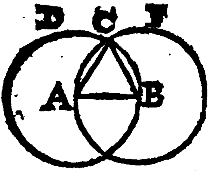
19. Every whole is equal to all its parts taken together.

20. If one whole be double to another, and that which is taken away from the firſt to that which is taken away from the ſecond, the remainder of the firſt ſhall be double to the remainder of the ſecond.

The Citations are to be underſtood in this manner; When you meet with two numbers, the firſt ſhews the Propoſition, the ſecond the Book; as by 4. 1. you are to underſtand the fourth Propoſition of the firſt Book; and ſo of the reſt. *Moreover, ax. denotes Axiome, poſt. Poſtulate, def. Definition, ſch. Scholium, cor. Corollary.*

PRO-

PROPOSITION I.



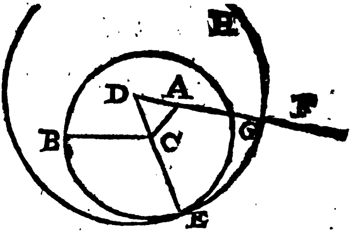
U*Pon a finite right line given AB, to deſcribe an equilateral triangle ACB.*

From the centers A and
B, at the diſtance of AB, or
BA, *a* deſcribe two circles a 3. *poſt.*
cutting each other in the
point C; from whence
b draw two right lines CA, CB. Then is AC *c* b 1. *poſt.*
= AB *c* = BC *d* = AC. *e* Wherefore the Tri- c 15. *def.*
angle ACB is equilateral. *Which was to be done.* d 1. *ax.*
e 23. *def.*

Scholium.

After the ſame manner upon the line AB may be deſcribed an Iſoſceles triangle, if the diſtances of the equal circles be taken greater or leſs than the line AB.

PROP. II.



At a point given A, to make a right line AG equal to a right line given BC.

From the center C, at the diſtance of CB, *a* de- a 3. *poſt.*
ſcribe the circle CBE. *b* join AC; upon which b 1. *poſt.*
c raiſe the equilateral triangle ADC. *d* Produce c 1. 1.
DC to E. From the center D, at the diſtance d 2. *poſt.*
of

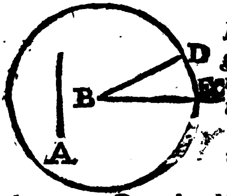
e 2. *poſt.* of DE, deſcribe the circle DEH; and let DA *e*
be produced to the point G in the circumference thereof. Then AG = CB.

f 15. *def.* For DG *f* = DE, and DA *g* = DC. Where-
g *conſtr.* fore AG *h* = CE *k* = BC *l* = AG. *Which was*
h 3. *ax.* *to be done.*
k 15. *def.* The putting of the point A within or without
l 1. *ax.* the line BC varies the caſes; but the conſtruction,
and the demonſtration, are every where alike.

Schol.

The line AG might be taken with a pair of compaſſes; but the ſo doing anſwers to no Poſtulate, as *Proclus* well intimates.

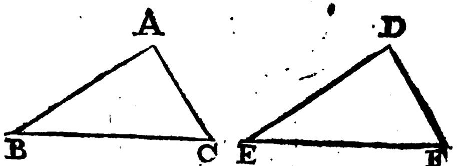
PROP. III.



Two right lines, A and BC, being given, from the greater BC to take away the right line BE equal to the leſſer A.

a 2. 1. At the point B *a* draw the right line BD = A. The circle deſcribed from the center B at the diſtance of BD ſhall cut off
b 15. *def.* BE *b* = BD *c* = A *d* = BE. *Which was to be done.*
c *conſtr.*
d 1. *ax.*

PROP. IV.

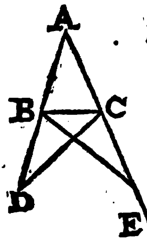


If two triangles BAC, EDF, have two ſides of the one BA, AC equal to two ſides of the other ED, DF, each to its correſpondent ſide (that is, BA = ED, and

and AC=DF) and have the angle A equal to the angle D contained under the equal right lines; they ſhall have the baſe BC equal to the baſe EF; and the triangle BAC ſhall be equal to the triangle EDF; and the remaining angles B, C, ſhall be equal to the remaining angles E, F, each to each, under which the equal ſides are ſubtended.

If the point D be applied to the point A, and
the right line DE plac'd upon the right line AB,
the point E ſhall fall upon B, becauſe DE *a*=AB, a *hyp.*
alſo the right line DF ſhall fall upon AC, becauſe
the angle A *a*= D. moreover the point F ſhall
fall upon the point C, becauſe AC *a* = DF. b 14. *ax.*
Therefore the right lines EF, BC ſhall agree,
becauſe they have the ſame Terms, and conſequently are equal. Wherefore the triangles
BAC, DEF, and the angles B, E, as alſo the
angles C, F, do agree, and are equal. *Which
was to be Demonſtrated.*

PROP. V.



The angles ABC, ACB, at the baſe of an Iſoſceles triangle ABC, are equal one to the other: And if the equal ſides AB, AC be produc'd, the angles CBD, BCE, under the baſe, ſhall be equal one to the other.

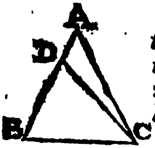
a Take AE=AD; and *b* join a 3. 1.
CD, and BE. b 1. *poſt.*

Becauſe, in the triangles ACD,
ABE, are AB *c* = AC, and AE c *hyp.*
d=AD, and the angle A common to them both, d *conſtr.*
e therefore is the angle ABE =ACD, and the angle e 4. 1.
AEB *e* = ADC, and the baſe BE *e*=CD; alſo
EC *f* = DB. Therefore in the triangles BEC, f 3. *ax.*
BDC *g* ſhall be the angle ECB = DBC. *Which* g 4. 1.
was to be Dem. Alſo therefore the angle EBC=
DCB. but the angle ABE *h*=ACD; therefore h *before.*
the angle ABC *k*=ACB. *Which was to be Dem.* k 3. *ax.*

Coroll.

Hence, every equilateral triangle is alſo equiangular.

PROP. VI.



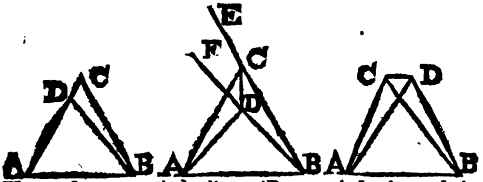
If two angles ABC, ACB of a triangle ABC be equal the one to the other, the ſides AC, AB ſubtended under the equal angles, ſhall alſo be equal one to the other.

If the ſides be not equal, let one
be bigger than the other, ſuppoſe BA ⊏ CA. *a*
a 3. 1. Make BD=CA, and *b* draw the line CD.
b 1. *poſt.* In the triangles DBC, ACB, becauſe BD *c*=
c *ſuppoſ.* CA, and the ſide BC is common, and the angle
d *hyp.* DBC *d*=ACB, the triangles DBC, ACB *e* ſhall
e 4. 1. be equal the one to the other, a part to the
f 9. *ax.* whole. *f Which is impoſſible.*

Coroll.

Hence, Every equiangular triangle is alſo equilateral.

PROP. VII.



Upon the ſame right line AB two right lines being drawn AC, BC, two other right lines equal to the former, AD, BD, each to each (viz. *AD = AC, and BD = BC*) *cannot be drawn from the ſame points A, B, on the ſame ſide C, to ſeveral points, as C and D, but only to C.*

1. *Caſe.* If the point D be ſet in the line AC,
a 9. *ax.* it is plain that AD is *a* not equal to AC.

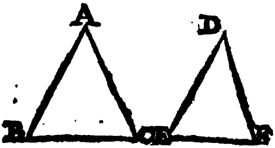
2. *Caſe.* If the point D be placed within the triangle ACB, then draw the line CD, and produce BDF, and BCE. Now you would have AD=AC.

then the angle ADC *b*=ACD; as alſo, becauſe b 5. 1.
BD *c* = BC, the angle FDC = *b* ACD. there- c *ſuppoſ.*
fore is the angle FDC $\sqsubset$ *d* ACD, that is, the d 9. *ax.*
angle FDC $\sqsubset$ ADC. *d Which is impoſſible.*

3. *Caſe.* If D falls without the triangle ACB, let CD be joined.

Again the angle ACD *e* = ADC, and the e 5. 1.
angle BCD *e* = BDC. *f* Therefore the angle f 9. *ax.*
ACD $\sqsubset$ BDC, *viz.* the angle ADC $\sqsubset$ BDC.
Which is impoſſible. Therefore, &c.

PROP. VIII.



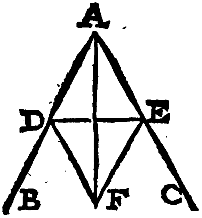
If two triangles ABC, DEF, have two ſides AB, AC equal to two ſides DE, DF, each to each, and the baſe BC equal to the baſe EF, then the angles contained under the equal right lines ſhall be equal, viz. *A to D.*

Becauſe BC *a*=EF, if the baſe BC be laid on a *hyp.*
the baſe EF, *b* they will agree: therefore whereas b *ax.* 8.
AB *c*=DE, and AC=DF, the point A will fall c *hyp.*
on D (for it cannot fall on any other point, by the precedent propoſition) and ſo the ſides of the angles A and D are coincident; *d* wherefore d 8. *ax.*
thoſe angles are equal. *Which was to be Dem.*

Coroll.

1. Hence, Triangles mutually equilateral, are alſo mutually *e* equiangular.
2. Triangles mutually equilateral, *e* are equal e 4. 1.
one to the other.

PROP. IX.



To biſect, or divide into two equal parts, a right-lined angle given BAC.

a 3. 1. *a* Take AD = to AE, and draw the line DE; upon which

b 1. 1. *b* make an equilateral triangle DFE. draw the right line AF; it ſhall biſect the angle.

c conſtr. For AD *c*= AE, and the ſide AF is common,

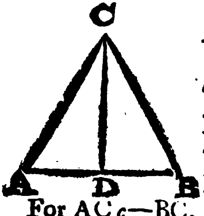
d 8. 1. and the baſe DF *c*= FE. *d* therefore the angle DAF = EAF. *Which was to be done.*

Coroll.

Hence it appears, how an angle may be cut into any equal parts, as 4, 8, 16, &c. to wit, by biſecting each part again.

The method of cutting angles into any equal parts required, by a Rule and Compaſs, is as yet unknown to Geometricians.

PROP. X.



To biſect a right line given AB.

Upon the line given AB

a 1. 1. *a* erect an equilateral triangle

b 9. 1. ABC; and *b* biſect the angle C with the right line CD. That line ſhall alſo biſect the line given AB.

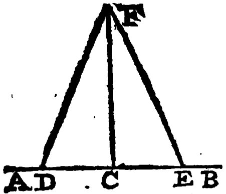
c conſtr. For AC *c*=BC, and the ſide CD is common,

d 4. 1. and the angle ACD *c*=BCD. therefore AD *d*=BD. *Which was to be done.*

The practice of this and the precedent Propoſition is eaſily ſhewn by the conſtruction of the 1 Prop. of this Book.

PROP.

PROP. XI.



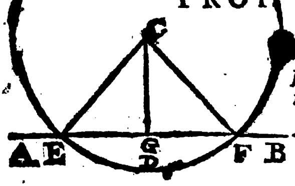
From a point C in a right line given AB to erect a right line CF at right angles.

a Take on either ſide of the point given CD a 3. 1.
= CE. upon the right line DE *b* erect an equi- b 1. 1.
lateral triangle, draw the line FC, and it will be the perpendicular required.

For the triangles DFC, EFC are mutually *c* e- c *conſtr.*
quilateral; *d* therefore the angle DCF=ECF. *e* d 8. 1.
therefore FC is perpendicular. *Which was to be done.* e 10. *def.*

The practice of this and the following is eaſily performed by the help of a ſquare.

PROP. XII.

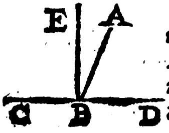


Upon an infinite right line given AB, from a point given that is not in it, to let fall a perpendicular right line CG.

From the center C *a* deſcribe a circle cutting the right line given
AB in the points E and F. Then biſect EF in a 3. *poſt.*
G, and draw the right line CG, which will be b 10. 1.
the perpendicular required.

Let the lines CE, CF be drawn. The triangles
EGC, FGC are mutually *c* equilateral. *d* there- c *conſtr.*
fore the angles EGC, FGC are equal, and by d 8. 1.
e conſequence right. *e* Wherefore GC is a per- e 10. *def.*
pendicular. *Which was to be done.*

PROP. XIII.



When a right line AB ſtanding upon a right line CD maketh angles ABC, ABD; it maketh either two right angles, or two angles equal to two right.

a *def.* 10. If the angles ABC, ABD be equal, *a* then they
make two right angles; if unequal, then from the
b 11. 1. point B *b* let there be erected a perpendicular BE.
c 19. *ax.* Becauſe the angle $ABC\ c =$ to a right $+ ABE$,
d 3. *ax.* and the angles $ABD\ d =$ to a right $- ABE$,
e 2. *ax.* therefore ſhall be $ABC + ABD\ e =$ to two
right angles $+ ABE - ABE =$ two right angles. *Which was to be demonſtrated.*

Corollaries.

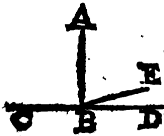
1. Hence, if one angle ABD be right, the other ABC is alſo right; if one acute, the other is obtuſe, and ſo on the contrary.

2. If more right lines than one ſtand upon the ſame right line at the ſame point, the angles ſhall be equal to two right.

3. Two right lines cutting each other make angles equal to four right.

4. All the angles made about one point, mak[illegible] four right; as appea[illegible] by Coroll.[illegible].

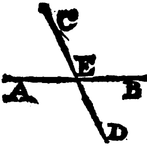
PR[illegible]R. XIV.



If to any right line AB, and a point therein B, two right lines, not drawn from the ſame ſide, do make the angles ABC, ABD on each ſide equal to two right, the lines CB, BD ſhall make one ſtrait line.

If you deny it, let CB, BE make one right line,
a 13. 1. then ſhall be the angle $ABC + ABE\ a =$ two right
b *hyp.* angles $b = ABC + ABD$. *Which is abſurd.*
c 9. *ax.*

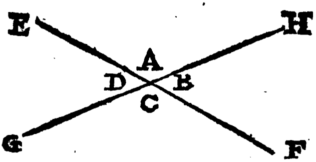
PROP. XV.



If two right lines AB, CD cut thro' one another, then are the two angles which are oppoſite, viz. CEB, AED, equal one to the other.

For the angle $AEC + CEB$
$a =$ to two right angles $= AEC$
a 13. 1. $+ AED$; *b* therefore $CEB = AED$. *Which was*
b 3. *ax.* *to be demonſtrated.*

Schol.

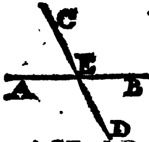


If to any right line GH, and in it a point A, two right lines being drawn EA, AF, and not taken on the ſame ſide, make the vertical (or oppoſite) angles D and B equal, thoſe right lines EA, AF, do meet directly and make one ſtrait line.

For two right angles are *a* equal to the angle D+A *a*=B+A. *b* therefore, EA, AF, are in a ſtrait line. *Which was to be demonſtrated.*

a 13. 1.
b 14. 1.

Schol.

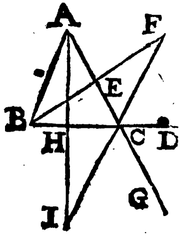


If four right lines EA, EB, EC, ED, proceeding from one point E, make the angles vertically oppoſite equal the one to the other, each two lines, AE, EB, and CE, ED, are placed in one ſtrait line.

For becauſe the angle AEC + AED + CEB + DEB *a* = to 4 right angles, therefore the angle AEC + AED *b* = CEB + DEB = to two right angles, *c* therefore CED and AEB are ſtrait lines. *Which was to be demonſtrated.*

a 4. c. 13. 1.
b *hyp.* and 2. *ax.*
c 14. 1.

PROP. XVI.



One ſide BC of any triangle ABC being produc'd, the outward angle ACD will be greater than either of the inward and oppoſite angles CAB, CBA.

Let the right lines AH, BE *a* biſect the ſides AC, BC; *b* produce EF = BE, and HI,

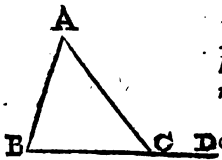
a 10. 1. & 1. *poſt.*
b 3. 1.

B

h = AH. and join FC, and IC; and produce ACG.

c *conſtr.* Becauſe CE *c* = EA, and EF *c* = EB, and the
d 15. 1. angle FEC *d* = BEA, the angle ECF *e* ſhall be
e 4. 1. equal to EAB. By the like argument is the an-
f 15. 1. gle ICH = ABH. Therefore the whole angle
g 9. *ax.* ACD (*f* BCG) *g* is greater than either the angle CAB or ABC. *Which was to be demonſtrated.*

PROP. XVII.

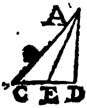


Two angles of any triangle ABC, which way ſoever they be taken, are leſs than two right angles.

Let the ſide BC be produced. Becauſe the angle ACD + ACB *a* = 2 right
a 13. 1. angles, and the angle ACD *b* $\sqsubset$ A, *c* therefore
b 16. 1. A + ACB $\sqsupset$ then two right angles. After the ſame
c 4. *ax.* manner is the angle B + ACB $\sqsupset$ then two right. Laſtly, the ſide AB being produced, the angle A + B will be alſo leſs than two right angles. *Which was to be demonſtrated.*

Coroll.

1. Hence it follows that in every triangle, wherein one angle is either right or obtuſe, the two others are acute angles.



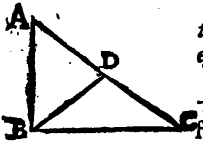
2. If a right line AE make unequal angles with another right line D, one acute AED, the other obtuſe AEC, a perpendicular AD let fall from any point A to the other line CD, ſhall fall on that ſide the acute is of.

For if AC, drawn on the ſide of the obtuſe angle, be a perpendicular, then in the triangle AEC
* 17. 1. ſhall AEC + ACE be* greater than two right angles. *Which is contrary to the precedent Prop.*

3. All the angles of an equilateral triangle, and the two angles of an Iſoſceles triangle that are upon the baſe, are acute.

PROP.

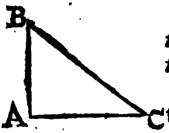
PROP. XVIII.



The greateſt ſide AC of every triangle ABC ſubtends the greateſt angle ABC.

From AC *a* take away AD a 3. 1.
= AB, and join BD. *b* There- b 5. 1.
fore is the angle ADB = ABD.
But ADB *c* ⊏ C; therefore is ABD ⊏ C; *d* there- c 16. 1.
fore the whole angle ABC ⊏ C. After the ſame d 9. *ax.*
manner, ſhall be ABC ⊏ A. *Which was to be dem.*

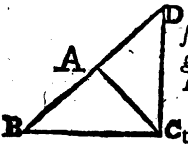
PROP. XIX.



In every triangle ABC, under the greateſt angle A is ſubtended the greateſt ſide BC.

For if AB be ſuppoſed equal
to BC, then will be the angle A
a = C, which is contrary to the
Hypotheſis: and if AB ⊏ BC, then ſhall be the an- a 5. 1.
gle C *b* ⊏ A, which is againſt the Hypotheſis.
Wherefore rather BC ⊏ AB; and after the ſame b 18. 1.
manner BC ⊏ AC. *Which was to be dem.*

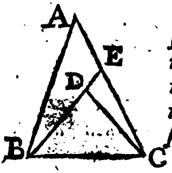
PROP. XX.



Of every triangle ABC two ſides BA, AC, any way taken, are greater than the ſide that remains BC.

Produce the line BA, *a* and a 3. 1.
take AD = AC, and draw the
line DC, *b* then ſhall the angle D be equal to b 5. 1.
ACD, *c* therefore is the whole angle BCD ⊏ D; c 9. *ax.*
d therefore BD (*e* BA + AC) ⊏ BC. *Which was* d 19. 1.
to be demonſtrated. e *conſtr.*
& 2. *ax.*

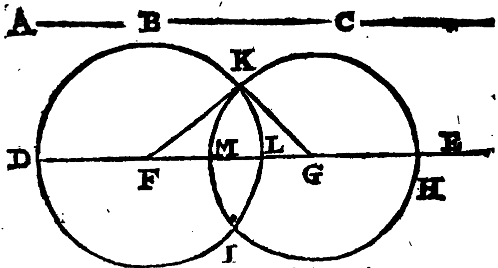
PROP. XXI.



If from the utmoſt points of one ſide BC of a triangle ABC two right lines BD, CD be drawn to any point within the triangle, then are both thoſe two lines ſhorter than the two other ſides of *the triangle, BA, CA; but do contain a greater angle, BDC.*

Let BD be produced to E. Then is CE + ED
a 20. 1. *a* $\sqsubset$ CD, and BD common to both, *b* then ſhall be
BD + DE + EC $\sqsubset$ CD + BD. Again, BA + AE
b 4. *ax.* *a* $\sqsubset$ BE. *b* therefore BA + AC $\sqsubset$ BE + EC. Wherefore 1. BA + AC $\sqsubset$ BD + DC. 2. The angle BDC
c 16. 1. *c* $\sqsubset$ DEC *c* $\sqsubset$ A. Therefore the angle BDC $\sqsubset$ A. *Which was to be demonſtrated.*

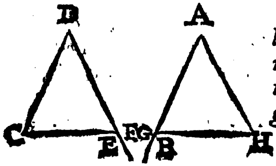
PROP. XXII.



To make a triangle FKG of three right lines FK, FG, GK, which ſhall be equal to three right lines given A, B, C. Of which it is neceſſary that any two taken together be longer than the third.

From the infinite line DE *a* take DF, FG, GH
a 3. 1. equal to the lines given, A, B, C. Then if from
the *b* centers F and G by the diſtances of FD
b 3. *poſt.* and GH, two circles be drawn cutting each
other in K, and the right lines KF, KG be join-
c 15. *def.* ed, the triangle FKG ſhall be made, *c* whoſe
ſides FK, FG, GK are equal to the three lines
d 1. *ax.* DF, FG, GH *d* that is to the three lines given
A, B, C. *Which was to be done.*

PROP. XXIII.

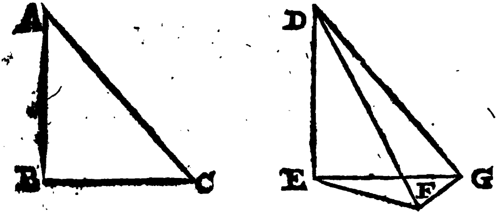


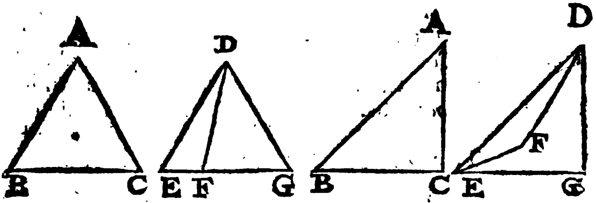
At a point A in a right line given AB to make a right-lined angle A equal to a right-lined angle given D.

a 1. *poſt.* Draw the right

line CF cutting the ſides of the angle given any
ways; *b* make AG = CD; upon AG *c* raiſe a trian- b 3. 1.
gle equilateral to the former CDF, ſo that AH be c 21. 1.
equal to DF, and GH to CF. then ſhall you have
the angle A *d* = D. *Which was to be done.* d 8. 1.

PROP. XXIV.





If two triangles ABC, DEF have two ſides of the one triangle AB, AC equal to two ſides of the other triangle DE, DF, each to other, and have the angle A greater than the angle EDF contained under the equal right lines, they ſhall alſo have the baſe BC greater than the baſe EF.

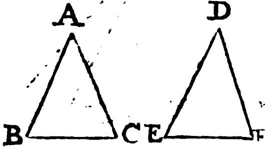
a Let the angle EDG be made equal to A, a 23. 1.
and the ſide DG *b* = DF *c* = AC; and let EG, b 3. 1.
and FG be joined. c *hyp.*

1. *Caſe.* If EG fall above EF; becauſe AB *d* = d *hyp.*
DE, and AC *e* = DG, and the angle A *e* = EDG, e *conſtr.*
f therefore is BC = EG. But becauſe DF *e* = DG, f 4. 1.
g therefore is the angle DFG = DGF; *h* therefore g 5. 1.
is the angle DFG ⊏ FFG, and by conſequence h 9. *ax.*
the angle EFG *h* ⊏ EGF, *k* wherefore EG (BC) k 19. 1.
⊏ EF.

2. *Caſe.* If the baſe EF falls in the ſame place
l 9. *ax.* with EG, *l* it is evident that EG (BC) $\sqsubset$ EF.

3. *Caſe.* If EG fall below EF, then becauſe DG
m 21. 1. + GE *m* $\sqsubset$ DF + FE, if from both DG, DF be
taken away, which are equal, EG (BC) remains
n 5. *ax.* *n* $\sqsubset$ EF. *Which was to be demonſtrated.*

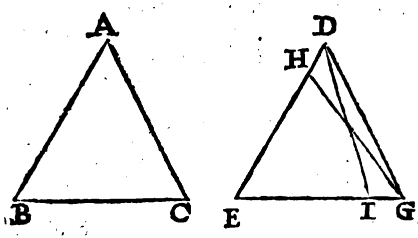
PROP. XXV.



If two triangles ABC, DEF have two ſides AB, AC equal to two ſides DE, DF, each to other, and have the baſe BC greater than the baſe EF, they ſhall alſo have the angle A contained under the equal right lines greater than the angle D.

a 4. 1. For if the angle A be ſaid to be equal to D, *a*
then is the baſe BC = EF, which is againſt the
b 24. 1. Hypotheſis. If it be ſaid the angle A $\sqsupset$ D, then *b*
will be BC $\sqsupset$ EF, which is alſo againſt the Hyp.
Therefore BC $\sqsubset$ EF. *Which was to be dem.*

PROP. XXVI.



If two triangles BAC, EDG have two angles of the one B, C equal to the two angles of the other E, DGE, each to his correſpondent angle, and have alſo one ſide of the one equal to one ſide of the other, either that ſide which lyeth betwixt the equal angles, or that which is ſubtended under one of the equal angles; the other ſides alſo

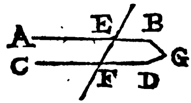
alſo of the one ſhall be equal to the other ſides of the other each to his correſpondent ſide, and the other angle of the one ſhall be equal to the other angle of the other.

1. *Hyp.* Let BC be equal to EG, which are the
ſides that lie between the equal angles. Then I
ſay BA=ED, and AC=DG, and the angle A=
EDG. For if it be ſaid that ED⊏BA, then *a* let EH a 3. 1.
be made equal to BA, and let the line GH be drawn.

Becauſe AB *b*=HE, and BC *c* = EG, and the b *ſuppoſ.*
angle B *c*=E, therefore ſhall be the angle EGH c *hyp.*
d = C *e* = DGE. *f Which is abſurd.* After the d 4. 1.
ſame manner let AC be equal to DG, *d* then will e *hyp.*
the angle A be equal to EDG. f 9. *ax.*

2. *Hyp.* Let AB be equal to ED. Then I ſay
BC=EG, and AC=DG, and the angle A=EDG.
For if EG be greater than BC, make EI=BC, and
join the line DI. Now becauſe AB *g*=ED, and g *hyp.*
BC *h* = EI, and the angle G *g* = E; therefore h *ſuppoſ.*
will be the angle EID *k*=C *l*=EGD. *m Which* k 4. 1.
is abſurd. Therefore is BC=EG, and ſo as be- l *hyp.*
fore AC=DG, and the angle A=EDG. *Which* m 16. 1.
was to be demonſtrated.

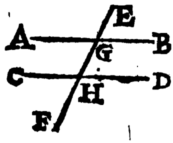
PROP. XXVII.



If a right line EF falling upon two right lines AB, CD, make the alternate angles AEF, DFE, equal the one to the other, then are the right lines AB, CD parallel.

If AB, CD be ſaid not to be parallel, produce
them till they meet in G. which being ſuppoſed,
the outward angle AEF will be *a* greater than a 16. 1.
the inward angle DFE, to which it was equal
by Hypotheſis. *Which is repugnant.*

PROP. XXVIII.

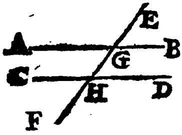


If a right line EF falling upon two right lines, AB, CD make the outward angle AGE of the one line equal to CHG the inward and oppoſite angle of *of the other on the ſame ſide, or make the inward angles on the ſame ſide AGH, CHG equal to two right angles, then are the right lines AB, CD parallel.*

Hyp. 1. Becauſe by Hypotheſis the angle AGE
a 25. 1. = CHG, *a* therefore are BGH, CHG alternate
b 27. 1. angles and equal: And *b* therefore are AB and
CD parallel.

Hyp. 2. Becauſe by Hypotheſis the angle AGH
a 13. 1. + CHG = to two right, *a* = AGH + BGH,
b 3. *ax.* *b* thence is CHG = BGH; and *c* therefore AB,
c 27. 1. CD are parallel. *Which was to be demonſtrated.*

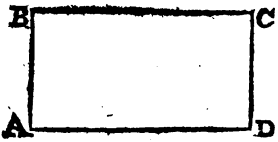
PROP. XXIX.



If a right line EF fall upon two parallels, AB, CD, it will both make the alternate angles DHG, AGH equal each to other, and the outward angle BGE equal to the inward and oppoſite angle on the ſame ſide DHE, as alſo the inward angles on the ſame ſide AGH, CHG equal to two right angles.

It is evident that AGH + CHG = 2 right an-
a 13. *ax.* gles; *a* otherwiſe AB,CD would not be parallel,
b 13. 1. which is contrary to the Hyp. But moreover
c 13. *ax.* the angle DHG + CHG *b* [illegible] 2 right; therefore is
d 15. 1. DHG [illegible] AHG *d* = BGE. *Which was to be dem.*

Coroll.

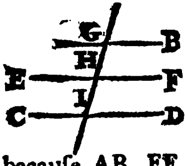


Hence it follows that every parallelogram AC having one angle right A, the reſt are alſo right.

a 29. 1. For A + B *a* = 2 right angles. Therefore
b 3. *ax.* whereas A is right, *b* muſt B be alſo right. By
the ſame argument are C and D right angles.

PROP.

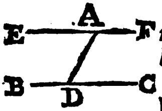
PROP. XXX.



Right lines (AB, CD) parallel to one and the ſame right line EF, are alſo parallel the one to the other.

Let GI cut the three right lines given any ways, then becauſe AB, EF are parallel, will be the angle
AGI *a* $= EHI$. alſo becauſe CD and EF are pa- a 29. 1.
rallel, will be the angle EHI *a* $= DIG$. *b* There- b 1. *ax.*
fore the angle $AGI = DIG$. *c* whence AB and c 27. 1.
CD are parallel. *Which was to be dem.*

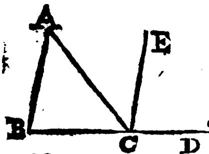
PROP. XXXI.



From a point given A to draw a right line AE parallel to a right line given BC.

From the point A draw a right line AD to any point of the given right line; with which at the point thereof
a A make an angle $DAE = ADC$. *b* then will AE a 23. 1.
and BC be parallel. *Which was to be done.* b 27. 1.

RROP XXXII.



Of any triangle ABC one ſide BC, being drawn out, the outward angle ACD ſhall be equal to the two inward oppoſite angles A, B, and the three inward angles of the triangle, A, B, ACB, ſhall be equal to two right angles.

From C *a* draw CE parallel to BA. Then is a 31. 1.
the angle A *b* $=$ ACE, and the angle B *b* $=$ ECD. b 29. 1.
Therefore $A + B$ *c* $ACE + ECD$ *d* $= ACD$. c 2. *ax.*
Which was to be demonſtrated. d 19. *ax.*
I affirm $ACD + ACB$ *e* $=$ two right angles; e 13. 1.
f therefore $A + B + ACB = 2$ right angles. *Which* f 1. *ax.*
was to be demonſtrated.

Coroll.

1. The three angles of any triangle taken together

ther are equal to three angles of any other triangle taken together. From whence it follows,

2. That if in one triangle, two angles (taken ſeverally, or together) be equal to two angles of another triangle (taken ſeverally, or together) then is the remaining angle of the one equal to the remaining angle of the other. In like manner, if two triangles have one angle of the one equal to one of the other, then is the ſum of the remaining angles of the one triangle equal to the ſum of the remaining angles of the other.

3. If one angle in a triangle be right, the other two are equal to a right. Likewiſe, that angle in a triangle which is equal to the other two, is it ſelf a right angle.

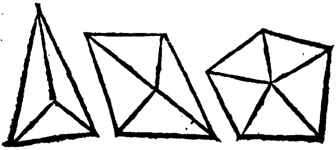
4. When in an Iſoſceles the angle made by the equal ſides is right, the other two upon the baſe are each of them half a right angle.

5. An angle of an equilateral triangle makes two third parts of a right angle. For $\frac{1}{3}$ of two right angles is equal to $\frac{2}{3}$ of one.

Schol.

By the help of this Propoſition you may know how many right angles the inward and outward angles of a right-lined figure make; as may appear by theſe two following Theorems.

THEOREM I.



All the angles of a right-lined figure do together make twice as many right angles, bating four, as there are ſides of the figure.

From any point within the figure, let right lines be drawn all thro' the angles of the figure, which ſhall reſolve the figure into as many triangles as there are ſides of the figure. Wherefore, whereas every triangle affords two right angles, all the triangles taken together will make up twice as many right angles as there are ſides. But the angles about the ſaid point within the figure make up four right; therefore, if from the angles of all the triangles you take away all the angles which are about the ſaid point, the remaining angles, which make up the angles of the figure, will make twice as many right angles, bating four, as there are ſides of the figure. *Which was to be dem.*

Coroll.

Hence, All right-lined figures of the ſame ſpecies have the ſums of their angles equal.

THEOREM II.

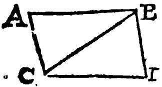
All the outward angles of any right-lined figure, taken together, make up four right angles.

For all the ſeveral inward angles of a figure with the ſeveral outward angles of the ſame make two right angles; therefore all the inward angles, together with the outward, make twice as many right angles as there are ſides of the figure; but (as it was now ſhewn) all the inward angles with four right, make twice as many right as there are ſides of the figure; therefore the outward angles are equal to four right angles. *Which was to be dem.*

Coroll.

All right-lined figures of whatſoever ſpecies have the ſums of their outward angles equal.

PROP. XXXIII.

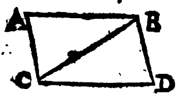


If two equal and parallel lines AB, CD be joined together with two other right lines AC, BD, then are thoſe lines alſo equal and parallel.

Draw a line from C to B. Now becauſe AB and CD are parallel, and the angle ABC *a* = BCD; a 29. 1.
and alſo by Hypotheſis AB=CD, and the ſide CB com-

b 4. 1. common, therefore is AC *b* = BD, and the an-
c 27. 1. gle ACB *b* = BD . *c* whence alſo AC, BD are parallel.

PROP. XXXIV.



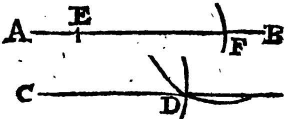
In parallelograms, as ABCD, the oppoſite ſides AB, CD, and AC, BD, are equal each to other; and the oppoſite angles A, D, and ABD, ACD are alſo equal; and the diameter BC biſects the ſame.

a *hyp.* Becauſe AB, CD *a* are parallel, *b* therefore is the angle ABC = BCD. Alſo becauſe AC, BD
b 29. 1. are *a* parallel, *b* therefore is the angle ACB =
c 2. *ax.* CBD; *c* therefore the whole angle ACD = ABD. After the ſame manner is A = D. Moreover becauſe the angles ABC, ACB lie at each end of the ſide CB, and are equal to BCD, CBD. *d* therefore
d 26. 1. is AC = BD, and AB *d* = CD, and ſo the triangle ABC = CBD. *Which was to be dem.*

Schol.

Every four-ſided figure ABDC having the oppoſite ſides equal, is a parallelogram.

a 27. 1. For by 8. 1. the angle ABC = BCD; *a* wherefore AB, CD are parallel. In like manner is the
b 35. *def.* 1. angle BCA = CBD; *a* wherefore AC, BD are alſo parallel. *b* Therefore ABCD is a parallelogram. *Which was to be demonſtrated.*

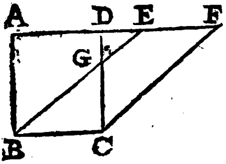


From hence may be learned how to draw a parallel CD to a right line given AB, thro' the point aſſigned C.

Take in the line AB any point, as E. From the centers E and C at any diſtance draw two equal circles EF, CD. From the center F by the ſpace of EC draw a circle FD, which ſhall cut the former circle CD in the point D. Then ſhall the line drawn CD be parallel to AB. for as it was before demonſtrated, CEFD is a parallelogram.

PROP.

PROP. XXXV.

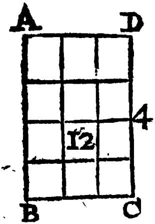


Parallelograms, BCDA, BCEF, which ſtand upon the ſame baſe BC, and between the ſame parallels AF, BC, are equal one to the other.

For AD *a* = BC *a* = a 34. 1.
EF, add DE common to both, *b* then is AE = b 2. ax.
DF. But alſo AB *a* = DC, and the angle A *c* = c 29. 1.
CDF. *d* Therefore is the triangle ABE = DCF. d 4. 1.
take away DGE common to both triangles, *e* e 3. ax.
then is the Trapezium ABGD = EGCF. add
BGC common to both, *f* then is the parallelo- f 2. ax.
gram ABCD = EBCF. *Which was to be dem.*

The demonſtration of any other caſes is not unlike, but much more plain and eaſy.

Schol.

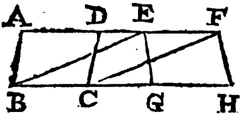


If the ſide AB of a right-angled parallelogram ABCD be conceived to be carried along perpendicularly thro' the whole line BC, or BC thro' the whole line AB, the Area or content of the Rectangle ABCD ſhall be produc'd by that motion. Hence a rectangle is ſaid to be made by the drawing or multiplication of two contiguous ſides. For example's ſake; let AB be ſuppoſed four foot, and BC be three: draw 3 into 4, there will be produced 12 ſquare feet for the Area of the Rectangle.

This being ſuppoſed, the dimenſion of any parallelogram (*EBCF) is found out by this Theorem. For the Area thereof is produced from the altitude BA drawn into the baſe BC. So the Area of the parallelogram AC = EBCF, is made by the drawing of BA into BC, therefore, &c.

* *See the figure of* Prop. 35.

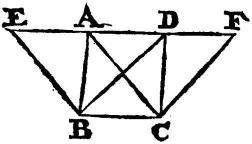
PROP. XXXVI.



Parallelograms, BCDA, GHFE, ſtanding upon equal baſes BC, GH, and betwixt the ſame parallels AF, BH, are equal to the other.

a *hyp.* Draw BE and CE. Becauſe BC $a = $ GH $b =$
b 34. 1. EF, *c* therefore is BCEF a parallelogram. Whence
c 33. 1. the parallelogram BCDA $d =$ BCFE $d =$ GHFE.
d 35. 1. *Which was to be dem.*

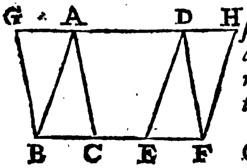
PROP. XXXVII.



Triangles, BCA, BCD, ſtanding upon the ſame baſe BC, and between the ſame parallels BC, EF, are equal one to the other.

a 31. 1. *a* Draw BE parallel to CA, *a* and CF parallel
b 34. 1. to BD. Then is the triangle BCA $b = \frac{1}{2}$ of the
c 35. 1. *and* parallelogram BCAE $c = \frac{1}{2}$ BDFC $b =$ BCD.
7. *ax.* *Which was to be dem.*

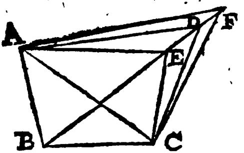
PROP. XXXVIII.



Triangles, BCA, EFD, ſet upon equal baſes BC, EF, and between the ſame parallels GH, BF, are equal the one to the other.

Draw BG parallel to CA and FH parallel to ED. Then is the trian-
a 34. 1. gle BCA $a = \frac{1}{2}$ Pgr. BCAG $b = \frac{1}{2}$ EDHF *c*
b 36. 1. *and* EFD. *Which was to be dem.*
7. *ax.* *Schol.*
c 34. 1. If the baſe BC be greater than EF, then is the triangle BAC ⊏ EDF, and ſo on the contrary.

PROP. XXXIX.



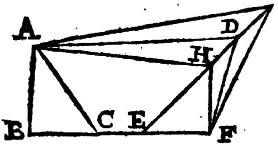
Equal triangles BCA, BCD, ſtanding on the ſame baſe BC, and on the ſame ſide, they are alſo between the ſame parallels AD, BC.

If you deny it, let another line AF be parallel to BC; and let CF be drawn. Then is the triangle CBF *a*= CBA *b*= CBD. *c Which is abſurd.* a 37. 1.

b *hyp.*

c 9. *ax.*

PROP. XL.



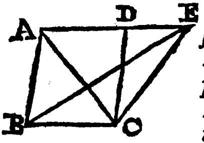
Equal triangles BCA, EFD, ſtanding upon equal baſes BC, EF, and on the ſame ſide, they are betwixt the ſame parallels.

If you deny it, let another line AH be parallel to BF, and let FH be drawn. Then is the triangle EFH *a*= BCA *b*= EFD. *c Which is abſurd.* a 38. 1.

b *hyp.*

c 9. *ax.*

PROP. XLI.



If a Pgr. ABCD have the ſame baſe BC with the triangle BCE, and be between the ſame parallels AE, BC then is the Pgr. ABCD double to the triangle BCE.

Let the line AC be drawn. Then is the triangle BCA *a* = BCE. therefore is the Pgr. ABCD. *b* = 2 BCA *c* = 2 BCE. *Which was to be dem.*

a 37. 1.

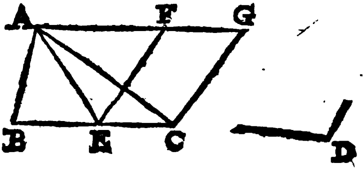
b 34. 1.

c 6. *ax.*

Schol.

From hence may the Area of any triangle BCE be found, for whereas the Area of the Pgr. ABCD is produced by the altitude drawn into the baſe, therefore ſhall the Area of a triangle be produced by half the altitude drawn into the baſe, or half the baſe drawn into the altitude, as if ſo be the baſe BC be 8, and the altitude 7, then is the Area of the triangle BCE, 28.

PROP. XLII.

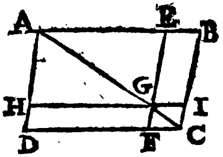


To make a Pgr. ECGF equal to a triangle given ABC in an angle equal to a right-lined angle given A.

a 31. 1. Through A *a* draw AG parallel to BC *b* make
b 23. 1. the angle BCG = D. *c* biſect the baſe BC in E,
c 10. 1. and draw EF parallel to CG. then is the problem reſolved.

For AE being drawn, the angle ECG is equal
d 31. 1. to D by conſtruction, and the triangle BAC *d*
e 41. 1. = 2 AEC *e* = Pgr. ECGF. *Which was to be done.*

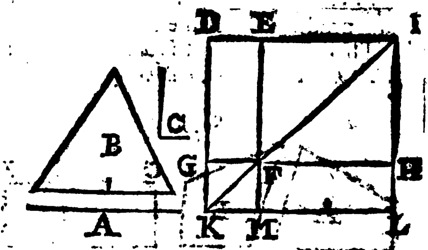
PROP. XLIII.



In every Pgr. ABCD, the complements DG, GB of thoſe Pgrs HE, FI, which ſtand about the diameter, are equal one to the other.

For the triangle ACD
a 34. 1. *a* = ACB, and the triangle AGH *a* = AGE, and
b 3. *ax.* the triangle GCF *a* = GCI. *b* Therefore the Pgr. DG = BG. *Which was to be dem.*

PROP. XLIV.

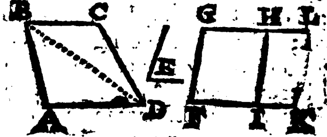


Upon a right line given A, to make a Parallelogram FL at a right-lined angle given C, equal to a triangle given B.

a Make a Pgr. FD = to the triangle B, ſo a 42. 1.
that the angle GFE may be equal to C. Produce GF till FH be equal to the line given A.
thro' H b draw IL parallel to EF, which let b 31. 1.
DE produced meet in the point I. let DG drawn forth meet with a right line drawn from I in the point K. thro' K b draw KL parallel to GH, with which let EF drawn out meet at M, and IH at L. Then ſhall FL be the Pgr. required.

For the Pgr. FL c = FD = B, d and the angle c 43. 1.
MFH=GFE=C. *Which was to be done.* d 15. 1.

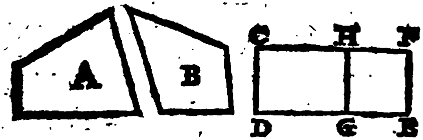
PROP. XLV.



Upon a right line given FG to make a Pgr. FL equal to a right-lined figure given ABCD, at a right-lined angle given E.

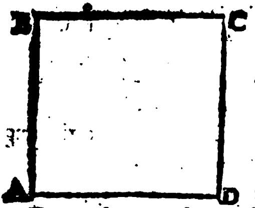
Reſolve the right-lined figure given into tri-
a 44. 1. angles BAD, BCD. then *a* make a Pgr. FH =
BAD, ſo that the angle F may be equal to E.
FI being produced, *a* make on HI the Pgr. IL =
b 19. *ax.* BCD. Then is the Pgr. FL *b* = FH + IL *c* =
c *conſtr.* ABCD. *Which was to be done.*

Schol.



Hence is eaſily found the exceſs, HE, whereby any right-lined figure, A, exceeds a leſs right-lined figure, B; namely, if to ſome right line, CD, both be applied, Pgr. DF = A, and DH = B.

PROP. XLVI.



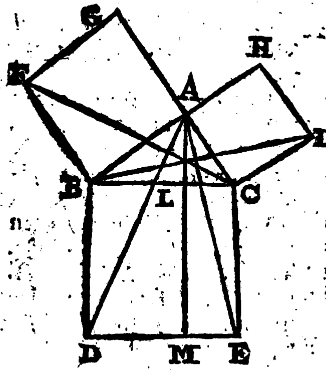
Upon a right line given AD to deſcribe a ſquare AC.

a 11. 3. *a* Erect two perpendiculars AB, DC, *b* equal
b 3. 1. to the line given AD; then join BC, and the thing required is done.

c *conſtr.* For, whereas the angle A + D *c* = 2 right, *d*
d 28. 1. therefore are AB, DC parallel. But they are alſo
e *conſtr.* *e* equal; *f* therefore AD, BC are both parallel
f 34. 1. and equal; therefore the figure AC is a Pgr.
g *ſch.*29.1. and equilateral. Moreover the angles are all
h 29. *def.* right, *g* becauſe one, A, is right; *h* therefore AC is a ſquare. *Which was to be done.*

After the ſame manner you may eaſily deſcribe a Rectangle contained under two right lines given.

PROP. XLVII.



In right angled Triangles; BAC, the ſquare BE, which is made of the ſide BC that ſubtends the right angle BAC is equal to both the ſquares BG, CH, which are made of the ſides AB, AC containing the right angle.

Join AE, and AD; and draw AM parallel to CE.

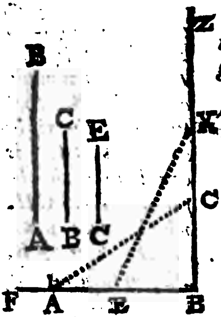
Becauſe the angle DBC *a* = FBA, add the a 12. *ax.*
angle ABC common to them both; then is the
angle ABD = FBC. Moreover AB *b* = FB, and
BD *b* = BC; *c* therefore is the triangle ABD = b 29. *def.*
FBC. But the Pgr. BM *d* = 2 ABD, and the c 4. 1.
Pgr. *d* BG = 2 FBC (for GAC is one right line d 41. 1.
by Hyp. and 14. 1.) *e* therefore is the Pgr. BM = e 6. *ax.*
BG. By the ſame way of argument is the Pgr.
CM = CH. Therefore is the whole BE = BG
+ CH. *Which was to be dem.*

Schol.

This moſt excellent and uſeful Theorem has deſerved the title of *Pythagoras*'s Theorem, becauſe he was the Inventer of it. By the help of which the addition and ſubſtraction of ſquares are performed; to which purpoſe ſerve the two following Problems.

PROBLEM I.

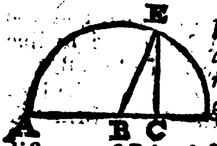
Andr. Tacq.



To make one ſquare equal to any number of ſquares given.

Let three ſquares be given whereof the ſides are AB, BC, CE. a 11. 1. *a* Make the right angle FBZ having the ſides infinite; and on them transfer BA and BC; join AC. then is ACq b 47. 1. *b* = ABq + BCq. Then transfer AC from B to X, and CE the third ſide given from B to E; join EX. *b* Then is EXq = EBq (CEq) + BXq (ACq) c 2. ax. *c* = CEq + ABq + BCq. *Which was to be done.*

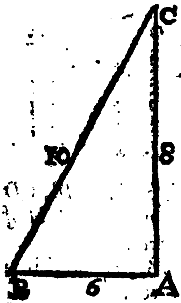
PROBLEM II.



Two unequal right lines being given AB, BC, to make a ſquare equal to the difference of the two ſquares of the given lines, AB, BC.

From the center B, at the diſtance of BA, deſcribe a circle; and from the point C erect a perpendicular CE meeting with a 47. 1. the circumference in E; and draw BE. *a* Then b 3. ax. is BEq (BAq) = BCq + CEq. *b* Therefore BAq − BCq = CEq. *Which was to be done.*

PROBLEM III.

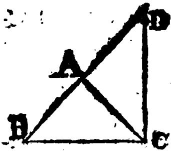


Any two ſides of a right-angled triangle ABC being known, to find out the third.

Let the ſides AB, AC encompaſſing the right angle, be, the one 6 foot, the other 8. Therefore, whereas $ACq + ABq = 64 + 36 = 100 = BCq$, thence is $BC = \sqrt{100} = 10$. 47. 1.

Otherwiſe, let the ſides AB, BC be known, the one ſix foot, the other 10. Therefore ſince $BCq - ABq = 100 - 36 = 64 = ACq$, thence 47. 1. is $AC = \sqrt{64} = 8$. *Which was to be dem.*

PROP. XLVIII.



If the ſquare made upon one ſide BC of a triangle be equal to the ſquares made on the other ſides of the triangle AB, AC, then the angle BAC comprehended under thoſe two other ſides of the triangle AB, AC, is a right angle.

Draw to the point A in AC a perpendicular line $DA = AB$, and join CD.

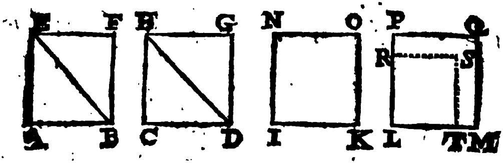
Now is *a* $CDq = ADq + ACq = ABq + ACq = BCq$. * Therefore is $CD = BC$. And therefore the triangles CAB, CAD are mutually equilateral. Wherefore the angle CAB *b* = CAD *c* = a right angle. *Which was to be dem.*

a 47. 1.
* *See the following Theor.*
b 8. 1.
c *conſtr.*

Schol.

We aſſumed in the demonſtration of the laſt Propoſition, $CD = BC$, becauſe CDq was equal to BCq: our aſſumption we prove by the following theorem.

THEOREME.



The ſquares AF, CG of equal right lines AB, CD are equal one to the other: And the ſides IK, LM of equal ſquares NK, PM, are equal one to the other.

1. *Hyp.* Draw the diameters EB, HD. Then it is evident that AF is *a* equal to the triangle EAB twice taken, and *b* equal to the triangle HCD twice taken, and equal to *a* CG. *Which was to be done.*

a 34. 1.
b 4. 1. & 6. ax.

2. *Hyp.* If it may be, let LM be greater than IK. Make LT = IK, and let LS be *a* equal to LTq. Therefore is LS *b* = NK *c* = LQ. *d Which is abſurd.*

a 46. 1.
b 1. part
c hyp.
d 9. ax.

Coroll.

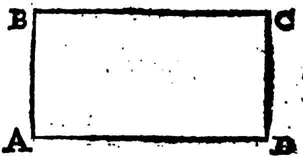
After the ſame manner any rectangles equilateral one to another, are demonſtrated alſo to be equal.

The End of the firſt Book.

THE

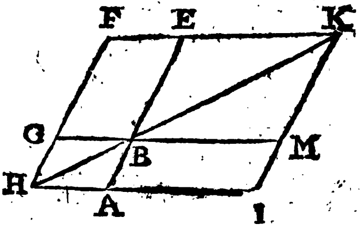
THE SECOND BOOK OF EUCLIDE's ELEMENTS.

Definitions.



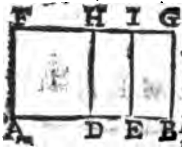
I. EVery right-angled Parallelogram ABCD is ſaid to be contained under two right lines AB, AD comprehending a right angle.

Therefore when you meet with ſuch as theſe, the rectangle under BA, AD, or for ſhortneſs ſake the rectangle BAD, or BA x AD (or ZA, for Z x A) the rectangle meant is that which is contained under the right lines BA, AD ſet at right angles.



II. In every Pgr. FHIK, any one of thoſe parallelograms which are about the diameter, together with its two complements is called a Gnomon. *As the Pgr. FB + BI + GA (EHM) is a Gnomon; and likewiſe the Pgr. FB + BI + EM (GKA) is a Gnomon.*

PROP. I.



If two right lines AF, AB, be given, and one of them AB divided into as many parts or ſegments as you pleaſe; the rectangle comprehended under the two whole right lines AB, AF, ſhall be equal to all the rectangles contained under the whole line AF and the ſeveral ſegments, AD, DE, EB.

a 11. 1. *a* Set AF perpendicular to AB. Thro' F *a* draw an infinite line FG perpendicular to AF. From the points, D, E, B erect perpendiculars DH, EI, BG. Then is AG a rectangle comprehended under AF, AB, and is *b* equal to the rectangles AH, DI, EG, that is (becauſe DH, EI, AF *c* are equal) to the rectangles under AF, AD, under AF, DE, under AF, EB. *Which was to be dem.*

b 19. *ax.* 1.

c 34. 1.

Schol.

If two right lines given be both divided into how many parts ſoever, the product of the whole multiplied into itſelf ſhall be the ſame with that of the parts multiplied into themſelves.

For let Z be $= A + B + C$, and $Y = D + E$; then, becauſe DZ *a* $= DA + DB + DC$, and EZ *a* $= EA + EB + EC$, and YZ *a* $= DZ + EZ$, *b* ſhall ZY be $= DA + DB + DC + EA + EB + EC$. *Which was to be dem.*

a 1. 2.

b 2. *ax.*

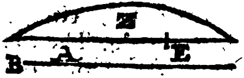
From hence is underſtood the manner of multiplying compounded right lines into compounded. For you muſt take all the Rectangles of the parts, and they will preſent you with the Rectangle of the wholes.

But whenſoever in the multiplication of lines into themſelves you meet with theſe ſigns − intermingled with theſe +, you muſt alſo have particular regard to the ſigns. For of + multiplied into − ariſeth −; but of − into − ariſeth +, *ex. gr.* let $+A$ be to be multiplied into $B - C$; then becauſe $+A$ is not affirmed of all B, but only of a part of it, whereby it exceeds C, therefore AC muſt remain denied; ſo that the product will be $AB - AC$. Or thus; becauſe B conſiſts of the parts C and $B-C$, *thence $AB = AC + A \times B - C$. * 1, 3, take away AC from either, then $AB - AC = A \times B - C$, In like manner if $-A$ be to be multiplied into $B-C$, then ſeeing by reaſon of the figure $-$ that A is not denied of all B. but only of ſo much as it exceeds C, therefore AC muſt remain affirmed. whence the product will be $-AB + AC$. Or thus; becauſe $AB = AC + A \times B - C$; take away all throughout, and there will be $-AB = AC - A \times -B + C$; add AC to either, and there will be $-AB + AC = A \times B - C$.

This being ſufficiently underſtood, the nine following Propoſitions, and innumerable others of that kind, ariſing from the comparing of lines multiplied into themſelves (which you may find done to your hand in *Vieta* and other Analytical Writers) are demonſtrated with great facility, by reducing the matter for the moſt part to almoſt a ſimple work.

Furthermore, *it appears that the product of * 19. ax. any magnitude multiplied into the parts of any number, is equal to the product of the ſame multiplied into the whole number: As $5A + 7A = 12A$, and $4A \times 5A + 4A \times 7A = 4A \times 12A$. Wherefore what is here deliver'd of the multiplying of right lines into themſelves, the ſame may be underſtood of the multiplying of numbers into themſelves, ſo that whatſoever is affirmed concerning lines in the nine following Theorems, holds good alſo in numbers; ſeeing they all immediately depend and are deriv'd from this firſt.

PROP. II.



If a right line Z be divided any-wiſe into two parts, the rectangles comprehended under the whole line Z and each of the ſegments A, E, are equal to the ſquare made of the whole line Z.

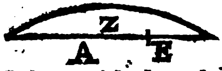
I ſay that $ZA + ZE = Zq$. For take $B = Z$;
a 1. 2. *a* then is $BA + BE = BZ$, that is (becauſe $B = Z$) $ZA + ZE = Zq$. *Which was to be dem.*

PROP. III.

If a right line Z be divided any-wiſe into two parts, the rectangle comprehended under the whole line Z and one of the ſegments E is equal to the rectangle made of the ſegments A, E, and the
a 1. 2. *ſquare deſcribed on the ſaid ſegment E.*

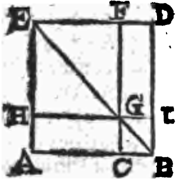
I ſay $ZE = AE + Eq$. *a* For $EZ = EA + Eq$.

PROP. IV.



If a right line Z be cut any-wiſe into two parts, the ſquare made of the whole line
a 3. 2. *Z is equal both to the ſquares made of the ſegments A, E, and to twice a rectangle made of the parts A, E.*

b 2. 2. I ſay that $Zq = Aq + Eq + 2\,AE$. For ZA *a* $=$
c 1. ax. $Aq + AE$, and ZE *a* $= Eq + EA$. Therefore whereas $ZA + ZE$ *b* $= Zq$, thence is $Zq = Aq + Eq + 2\,AE$. *Which was to be dem.*



Otherwiſe thus; Upon the right line AB make the ſquare AD, and draw the diameter EB; through C, the point wherein the line AB is divided, draw the perpendicular CF; and thro' the point G draw HI parallel to AB.

Becauſe the angle $EHG = A$ is a right angle,
d 4. cor. 32. 1. and AEB is half a right, *e* therefore is the remaining angle HGE half a right angle. There-
e 32. 1. fore is HE *f* $=$ HG *g* $=$ EF *g* $=$ AC, ſo that HF
f 6. 1. *h* is the ſquare of the right line AC. After the
g 34. 1. ſame manner is CI proved to be CBq. Therefore
h 29. def. AG, GD are rectangles under AC, CB. where-
k 19. ax. 1. fore the whole ſquare AD *k* $= ACq + CBq + 2\,ACB$. *Which was to be dem.*

Coroll.

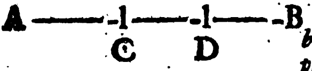
Coroll.

1. Hence it appears that the Parallelograms which are about the diameter of a ſquare are alſo ſquares themſelves.

2. That the diameter of any ſquare biſects its angles.

3. That if $A = \frac{1}{2} Z$, then $Zq = 4\,Aq$, and $Aq = \frac{1}{4} Zq$. As on the contrary, if it be ſo that $Zq = 4\,Aq$, then is $A = \frac{1}{2} Z$.

PROP. V.



If a right line AB be cut into equal parts AC, CB, and into unequal parts, AD, DB, the rectangle comprehended under the unequal parts AD, BD, together with the ſquare that is made of the intermediate part CD, is equal to the ſquare that is made of the half line CB.

I ſay that $CBq = ADB + CDq$.

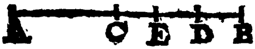
For theſe are all equal. $\left\{\begin{array}{l} CBq. \\ {}^{a}\,CDq+CDB+DBq+CDB. \\ CDq+{}^{b}\,CBD\,({}^{c}\,AC\times BD)+CDB. \\ CDq+{}^{d}\,ADB. \end{array}\right.$

a 4. 2.
b 3. 2.
c *hyp.*
d 1. 2.

This Theorem is ſomewhat differently expreſs'd, and more eaſily demonſtrated thus; *A Rectangle made of the ſum and the difference of two lines A, E, is equal to the difference of their ſquares.*

For if $A + E$ be multiplied into $A - E$, there ariſeth $Aq - AE + EA - Eq = Aq - Eq$. *Which was to be dem.*

Schol.



If the line AB be divided otherwiſe, (*viz.*) nearer to the point of biſection, in E; Then is $AEB \sqsubset ADB$.

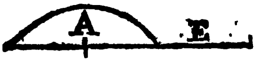
For if $AEB \,{}^{a}\!= CBq - CEq$. and $ADB \,{}^{a}\!= CBq$ a 5. 2. *and*
$- CDq$. Therefore, whereas $CDq \sqsubset CEq$, thence 3. *ax.*
is $AEB \sqsubset ADB$. *Which was to be dem.*

Coroll.

1. Hence is $ADq + DBq \sqsubset AEq + EBq$. For b 4. 2. $ADq + DBq + 2\,ADB$ *b* $= ABq$ *b* $= AEq + EBq + 2\,AEB$. Therefore becauſe $2\,AEB \sqsubset 2\,ADB$. thence is $ADq + DBq \sqsubset AEq + EBq$. *Which was to be dem.*

c 3. ax. 2. Hence is $ADq + DBq - AEq +$ *c* $+ EBq \sqsubset 2\,AEB - 2\,ADB$.

P R O P. VI.



If a right line A be divided into two equal parts, and another right line E added to the ſame directly in one right line, then the rectangle comprehended under the whole and the line added, (viz. A + E) and the line added E, together with the ſquare which is made of $\frac{1}{2}$ the line A, is equal to the ſquare of $\frac{1}{2}$ A+E taken as one line.

a 4. & 3. Cor. 4. 2. I ſay that $\frac{1}{4}Aq$ (*a* $Q{:}\,\frac{1}{2}A$) $+ AE + Eq = Q{:}\,\frac{1}{2}A + E$. *a* For, $Q\,\frac{1}{2}A + E = \frac{1}{4}Aq + Eq + AE$.

Coroll.

Hence it follows that if 3 right lines E, $E + \frac{1}{2}A$, $E + A$ be in Arithmetical proportion, then the Rectangle contained under the extreme terms E, $E + A$, together with the ſquare of the difference $\frac{1}{2}A$, is equal to the ſquare of the middle term $E + \frac{1}{2}A$.

P R O P. VII.

If a right line Z be divided any-wiſe into two parts, the ſquare of the whole line Z together with the ſquare made of one of the ſegments E, is equal to a double rectangle comprehended under the whole line Z and the ſaid ſegment E, together with the ſquare made of the other ſegment A.

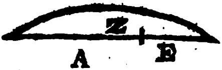
a 4. 2. b 3. 2. I ſay that $Zq + Eq = 2\,ZE + Aq$. For Zq *a* $= Aq + Eq + 2\,AE$. and $2\,ZE$ *b* $= 2\,Eq + 2\,AE$. *Which was to be dem.*

Coroll.

Hence it follows that the ſquare of the difference of any two lines Z, E, is equal to the ſquares of both the lines leſs by a double rectangle comprehended under the ſaid lines.

c For $Zq + Eq - 2\,ZE = Aq = Q: Z - E$. c 7. 2. and 3. *ax.*

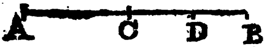
PROP. VIII.



If a right line Z be divided any-wiſe into two parts, the rectangle comprehended under the whole line Z and one of the ſegments E four times, together with the ſquare of the other ſegment A, is equal to the ſquare of the whole line Z and the ſegment E taken as one line Z + E.

I ſay that $4\,ZE + Aq = Q: Z + E$. For
$2\,ZE$ *a* $= Zq + Eq - Aq$. Therefore $4\,ZE +$ a 7. 2. *and*
$Aq = Zq + Eq + 2\,ZE$ *b* $= Q: Z + E$. *Which* 3. *ax.*
was to be dem. b 4. 1.

PROP. IX.



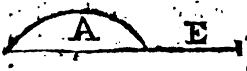
If a right line AB be divided into equal parts AC, CB, and into unequal parts, AD, DB, then are the ſquares of the unequal parts AD, DB, together, double to the ſquare of the half line AC, and to the ſquare of the difference CD.

I ſay that $ADq + DBq = 2\,ACq + 2\,CDq$. For
$ADq + DBq$ *a* $= ACq + CDq + 2\,ACD + DBq$. a 4. 1.
But $2\,ACD$ (*b* $2\,BCD$) $+ DBq$ *c* $= CBq$ (ACq) b *hyp.*
$+ CDq$. *d* Therefore $ADq + DBq = 2\,ACq +$ c 7. 2.
$2\,CDq$. *Which was to be dem.* d 2. *ax.*

This may be otherwiſe deliver'd and more eaſily demonſtrated thus; *The aggregate of the ſquares made of the ſum and the difference of two right lines A, E, is equal to the double of the ſquares made from thoſe lines.*

For $Q: A + E$ *a* $= Aq + Eq + 2\,AE$. and $Q: A$ a 4. 2.
$- E$ *b* $= Aq + Eq - 2\,AE$. Theſe added toge- b *ſch.* 7. 2.
ther make $2\,Aq + 2\,Eq$. *Which was to be dem.*

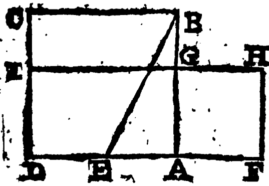
PROP. X.



If a right line A be divided into two equal parts, and another line be added in a right line with the ſame, then is the ſquare of the whole line, together with the added line, (as being one line,) together with the ſquare of the added line E, double to the ſquare of $\frac{1}{2}$ A and the added line E, taken as one line.

a 4. 2. I ſay that Eq+Q: A+E, *i. e.* *a* Aq+ 2 Eq + 2
b *cor.* 4.2. AE = 2 Q: ^{1} A + 2 Q: $\frac{1}{2}$ A+E. For 2 Q: $\frac{1}{2}$
c 4. 2. A *b* = $\frac{1}{2}$ Aq. And 2 Q: $\frac{1}{2}$ A + E *c* = $\frac{1}{2}$ Aq + 2 Eq + 2 AE. *Which was to be dem.*

PROP. XI.



To cut a right line given AB in point G, ſo that the rectangle comprehended under the whole line AB, and one of the ſegments BG ſhall be equal to the ſquare that is made of the other ſegments AG.

a 46. 1. Upon AB *a* deſcribe the ſquare AC. *b* Biſect
b 10. 1. the ſide AD in E, and draw the line EB; from the line EA produced take EF = EB. On AF make the ſquare AH. Then is AH = AB × BG.

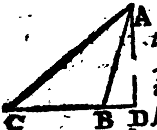
For HG being drawn out to I; the rectan-
c 6. 2. gle DH + EAq *c* = EFq *d* = EBq *e* = BAq + EAq.
d *conſtr.* Therefore is DH *f* = BAq = to the ſquare AC.
e 47. 1. Take away AI common to both, then remains
f 3. *ax.* the ſquare AH = GC, that is, AGq = AB × BG. *Which was to be done.*

Schol.

This Propoſition cannot be performed by
* 6. 13. numbers; * for there is no number that can be ſo divided, that the product of the whole into one part ſhall be equal to the ſquare of the other part.

PROP.

PROP. XII.



In obtuſe-angled triangles, ABC, the ſquare that is made of the ſide AC ſubtending the obtuſe angle ABC is greater than the ſquares of the ſides BC, AB that contain the obtuſe angle ABC, by a double rectangle contained under one of the ſides BC, which are about the obtuſe angle ABC, on which ſide produced the perpendicular AD falls, and under the line BD, taken without the triangle from the point on which the perpendicular AD falls to the obtuſe angle ABC.

I ſay that $ACq = CBq + ABq + 2\,CB \times BD$.

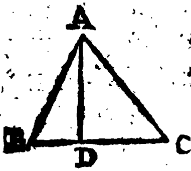
$$\text{For theſe are all equal.}\left\{\begin{array}{ll} ACq. & \\ {}^{a}\,CDq + ADq. & a\ 47.\ 1. \\ {}^{b}\,CBq + 2\,CBD + BDq + ADq. & b\ 4.\ 1. \\ CBq + 2\,CBD + {}^{c}\,ABq. & c\ 47.\ 1. \end{array}\right.$$

Scholium.

Hence, *the ſides of any obtuſe angled triangle ABC being known, the ſegment BD intercepted betwixt the perpendicular AD and the obtuſe angle ABC, as alſo the perpendicular itſelf AD ſhall be eaſily found out.*

Thus. Let AC be 10, AB 7, CB 5. Then is ACq 100, ABq 49, CBq 25. And ABq + CBq = 74. Take that out of 100, then will 26 remain for 2 CBD. Wherefore CBD ſhall be 13; divide this by CB 5, there will $2\frac{3}{5}$ be found for BD. Whence AD will be found out by the 47. 1.

PROP. XIII.



In acute-angled triangles ABC, the ſquare made of the ſide AB ſubtending the acute angle ACB, is leſs than the ſquares made of the ſides AC, CB comprehending the acute angle ACB by a double rectangle contained under one of the ſides BC, which are about the acute angle ACB on which the perpendicular AD falls, and under the line DC taken within the triangle from the perpendicular AD to the acute angle ACB.

I ſay that ACq + BCq = ABq + 2 BCD.

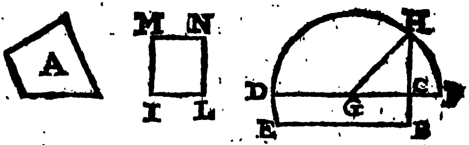
For theſe are equal.
ACq + BCq.
a 47. 1. *a* ADq + DCq + BCq.
b 7. 2. *b* ADq + BDq + 2 BCD.
c 47. 1. *c* ABq + 2 BCD.

Coroll.

Hence, *The ſides of an acute-angled triangle ABC being known, you may find out the ſegment DC intercepted betwixt the perpendicular AD and the acute angle ACB, as alſo the perpendicular it ſelf AD.*

Let AB be 13, AC 15, BC 14. Take ABq (169) from ACq + BCq, that is, from 225 + 196 = 421. Then remains 252 for 2 BCD. wherefore BCD will be 126. divide this by BC 14, then will 9 be found out for DC. From whence it follows AD = √: 225 − 81 = 12.

PROP. XIV.



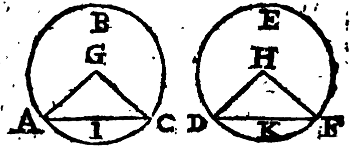
To find a ſquare ML equal to a right-lined figure given A.

a 45. 1. *a* Make the rectangle DB = A, and produce the greater ſide thereof DC to F, ſo that CF = CB.
b 10. 1. *b* biſect DF in G, about which as the center at the diſtance of GF deſcribe the circle FHD, and draw out CB till it touch the circumference in
* 46. 1. H. Then ſhall be CHq = * ML = A.
c *conſtr.* For let GH be drawn. Then is A *c* = DB *c*
d 5. 2. *and* = DCF *d* = GFq − GCq *e* = HCq *e* = ML.
3. *ax.* *Which was to be done.*
e 47. 1. *and*

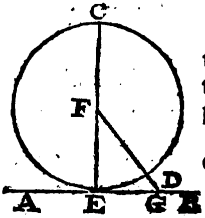
The End of the ſecond Book.

THE THIRD BOOK OF EUCLIDE's ELEMENTS.

Definitions.

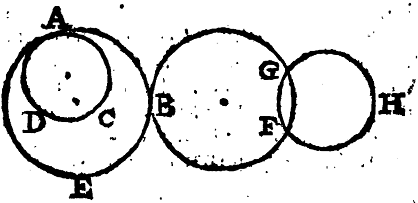


I. EQual circles (GABC, HDEF) are ſuch whoſe diameters are equal; or, from whoſe centers right lines drawn GA, HD, are equal.



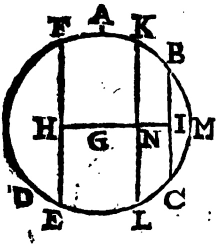
II. A right line AB is ſaid to touch a circle FED, when touching the ſame, and being produced, it cutteth it not.

The right line FG cuts the circle FED.

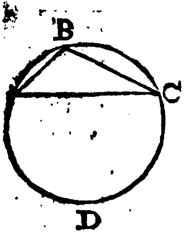


III. Circles DAC, ABE (and alſo FBG, ABE) are ſaid to touch one to the other, which touch, but cut not one the other.

The circle BFG cuts the circle FCH



IV. In a circle GABD, right lines FE,KL are ſaid to be equally diſtant from the center, when perpendiculars GH, GN drawn from the center G to them are equal. And that line BC is ſaid to be furtheſt diſtant from it, on whom the greater perpendicular GI falls.

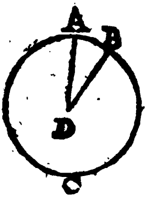


V. A ſegment of a circle (ABC) is a figure contained under a right line AC, and a portion of the circumference of a circle ABC.

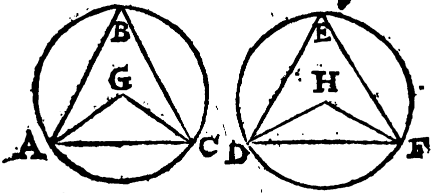
VI. An angle of a ſegment CAB, is that angle which is contained under a right line CA and an arch of a circle AB.

VII. An angle ABC is ſaid to be in a ſegment ABC, when in the circumference thereof ſome point B is taken, and from it right lines AB, CB, drawn to the ends of the right line AC, which is the baſe of the ſegment; then the angle ABC contained under the adjoined lines AB, CB, is ſaid to be angle in a ſegment.

VIII. But when the right lines AB, BC comprehending the angle ABC, do receive any periphery of the circle ADC, then the angle ABC is ſaid to ſtand upon that periphery.

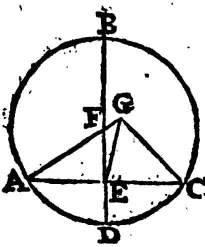


IX. A ſector of a circle (ADB) is when an angle ADB is ſet at the center D of that circle; namely, that figure ADB, comprehended under the right lines AD, BD containing the angle, and the part of the circumference received by them AB.



X. Like ſegments of a circle (ABC, DEF) are thoſe which include equal angles (ABC, DEF;) or, in whom the angles ABC, DEF are equal.

PROP. I.



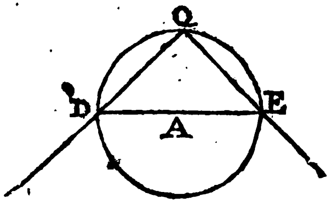
To find the center F of a circle given ABC.

Draw a right line AC any-wiſe in the circle, which biſect in E, thro' E draw a perpendicular DB; and biſect the ſame in F, the point F ſhall be the center.

If you deny it, let G a point without the line DB be the center, (for it cannot be in the line BD, ſince that cannot be divided equally in any point but F;) let the lines GA, GC GE drawn: Now if G be the center, *a* then is GA=GC, and AE= a 15.*def.*1.
EC by conſtruction, and the ſide GE common. *b* b 8. 1.
Therefore are the angles GEA, GEC equal, and c 10.*def.*1.
c conſequently right. *d* Therefore the angle d 12. *ax.*
GEC=FEC; *e Which is abſurd.* e 9. *ax.*

Coroll.

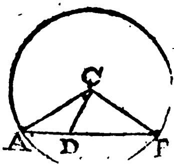
Hence, if a right line BD biſect any right line AC in a circle at right angles, the center ſhall be in the line BD that cuts the other.



Andr. Tacq.

The center of a circle is eaſily found out by applying the top of a ſquare Q to the circumference thereof. For if the right line DE that joins the points D, E, in which the ſides of the ſquare QD, QE cut the circumference, be biſected in A, the point A ſhall be the center. The demonſtration whereof depends upon Prop. 31. of this Book.

PROP. II.



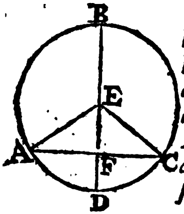
If in the circumference of a circle CAB any two points A, B be taken, the right line AB which joins thoſe two points ſhall fall within the circle.

Take in the right line AB any point D; from the center
a 15. *def.* 1. C draw CA, CD, CB. Becauſe CA *a* = CB,
b 5. 1. therefore is the angle A *b* = B. But the angle
c 16. 1. CDB *c* ⊏ A, therefore is CDB ⊏ B, therefore
d 19. 1. CB *d* ⊏ CD. But CB only reaches the circumference, therefore CD comes not ſo far; wherefore the point D is within the circle. The ſame may be proved of any other point in the line AB. And therefore the whole line AB falls within the circle. *Which was to be dem.*

Coroll.

Hence, if a right line touch a circle, ſo that it cut it not, it touches but in one point.

PROP. III.



If in a circle EABC a right line BD drawn thro' the center, biſect any other line AC not drawn through the center, it ſhall alſo cut it at right angles in F: And if it cuts it at right angles, it ſhall alſo biſect the ſame.

From the center E let the lines EA, EC be drawn.

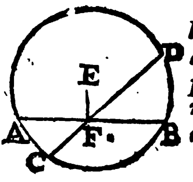
1. *Hyp.* Becauſe AF *a* = FC, and EA *b* = a *hyp.*
EC, and the ſide EF common; the angles EFA, b 15.*def.*1.
EFC *c* ſhall be equal, and *d* conſequently right. c 8.1.
Which was to be dem. d 10.*def.*1.

Hyp. 2. Becauſe EFA *e* = EFC, and the angle e *hyp. and*
EAF *f* = ECF, and the ſide EF common; *g* 12. *ax.*
therefore is AF=FC. Therefore is AC cut into f 5.1.
two equal parts. *Which was to be dem.* g 26.1.

Coroll.

Hence, in any equilateral or Iſoſceles triangle, if a line drawn from the vertical angle biſect the baſe, that line is perpendicular to it. And on the contrary, a perpendicular drawn from the vertical angle biſects the baſe.

PROP. IV.



If in a circle ACD two right lines AB, CD cut thro' one another, yet neither of them paſs thro' the center E, then neither of thoſe lines are divided into equal parts.

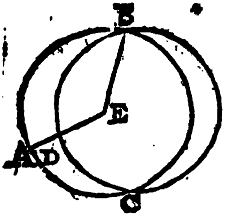
For if one line paſs thro'

the center, it appears that it cannot be biſected by the other; becauſe by Hypotheſis, the other does not paſs thro' the center.

If neither of them paſs thro' the center, then from the center E draw EF: now if AB, CD were both biſected in F, then *a* would the angles EFB, EFD be both right, and conſequently equal. *b Which is abſurd.*

a 3. 3.
b 9. ax.

PROP. V.

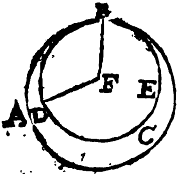


If two circles BAC, BDC cut one the other, they ſhall not have the ſame center E.

For otherwiſe the lines EB, EDA drawn from E the common center, would DE be *a* = EB *a* = EA. *b Which is abſurd.*

a 15. def. 1.
b 9. ax.

PROP. VI.



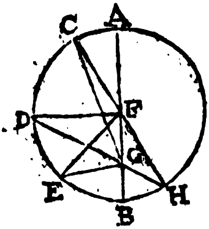
If two circles BAC, BDE, inwardly touch one the other (in B) *they have not one and the ſame center F.*

For otherwiſe the right lines FB, FDA drawn from the center F, would be FD *a* = FB *a* = FA. *b Which is abſurd.*

a 15. def. 1.
b 9. ax.

PROP.

PROP. VII.



If in AB the diameter of a circle, ſome point G be taken, which is not the center of the circle, and from that point certain right lines GC, GD, GE fall on the circle, the greateſt line ſhall be that (GA) in which is the center F; the leaſt, the remainder of the ſame line (GB) And of all the other lines, the line GC neareſt to that which was drawn thro' the center is always greater than any line farther removed GD; and only two lines are equal GE, GH, which fall upon the circle from the ſame point, on each ſide of the leaſt GB or of the greateſt GA.

From the center F draw the right lines FC,
FD, FE; and *a* make the angle BFH=BFE. a 23. 1.

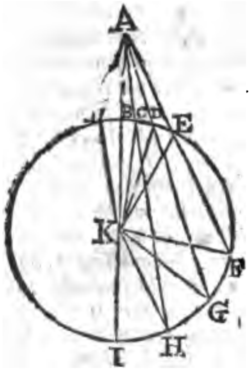
1. GF + FC (that is GA) *a* ⊏ GC. *Which* a 20. 1.
was to be dem.

2. The ſide FG is common, and FC *b* = FD, b 15. *def.* 1.
and the angle GFC *c* ⊏ GFD; *d* wherefore the c 9. *ax.*
baſe GC ⊏ GD. d 24. 1.

3. FB (FE) *e* ⊐ GE + GF. *f* Therefore FG, e 20.
which is common, being taken away from both, f 5. *ax.*
BG ⊐ EG.

4. The ſide FG is common, and FE=FH, and
the angle BFH *g* = BFE; *h* Therefore is GE= g *conſtr*
GH. But that no other line GD from the point h 4. 1.
G can be equal to GE, or GH, is already proved. *Which was to be dem.*

PROP. VIII.



If ſome point A be taken without a circle, and from that point be drawn certain right lines AI, AH, AG, AF to the circle, and of thoſe one AI be drawn through the center K, and the others any-wiſe; of all thoſe lines that fall on the concave of the circumference, that is the greateſt AI which is drawn through the center; and of the others, that which is neareſt (AH) to the line that paſſes through the center is greater than that which is more diſtant AG. But of all thoſe lines that fall on the convex part of the circle, the leaſt is that AB which is drawn from the point A to the diameter IB; and of the others, that (AC) which is neareſt to the leaſt, is leſs than that which is farther diſtant AD. And from that point there can be only two equal right lines AC, AL drawn, which ſhall fall on the circumference on each ſide of the leaſt line AB or of the greateſt AI.

From the center K draw the right lines KH, KG, KF, KC, KD, KE. and make the angle $AKL = AKC$.

a 20. 1. 1. $AI\ (AK + KH)$ *a* $\sqsubset AH$.

2. The ſide AK is common, and $KH = KG$,
b 24. 1. and the angle $AKH \sqsubset AKG$; *b* therefore the baſe $AH \sqsubset AG$.

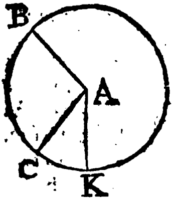
c 20. 1. 3. KA *c* $\sqsupset KC + CA$. From hence take away KC, KB that are equal; then will remain
d 5. ax. AB *d* $\sqsupset AC$.

e 21. 1. 4. $AC + CK$ *e* $\sqsupset AD + DK$. From thence take away CK, DK that are equal; then re-
f 5. ax. mains AC *f* $\sqsupset AD$,

5. The ſide KA is common, and KL = KC, and the angle AKL *g* = AKC; *h* therefore LA = *a* CA. But that no other line could be drawn equal to theſe, was proved above. Therefore, *&c.*

g *conſtr.*
h 4. 1.

PROP. IX.

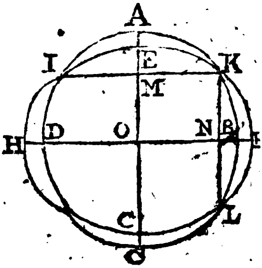


If in a circle BCK a point A be taken, and from that point more than two equal right lines AB, AC, AK, drawn to the circumference, then is that point A the center of the circle.

For *a* from no point without the center can more than two right lines equal be drawn to the circumference. Therefore A is the center. *Which was to be dem.*

a 7. 3.

PROP. X.



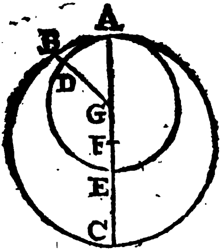
A circle I A K B L cannot cut another circle I E K F L in more than two points.

Let one circle, if it may be, cut the other in three points, I, K, L, and I K, K L being join'd, let them be biſected in M and N. *a* Both circles have their centers in their perpendiculars MC, NH, and in the interſection of thoſe perpendiculars, which is O. *b* Therefore the circles that cut each other have the ſame center. Which is falſe, by Prop. 5. 3.

a *cor.* 1. 3.
b 5. 3.

PROP.

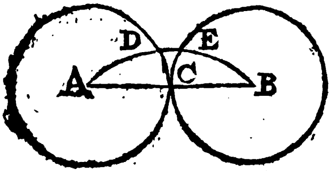
PROP. XI.



If two circles GADE, FABC touch one the other inwardly, and their centers be taken G, F; a right line FG joining their centers, and produced, ſhall cut the circumference in A the point of contact of the circles.

If it can be, let the right line FG produced cut the circles in ſome other point than A; ſo that not FGA, but FGDB ſhall be a right line.
a 15. *def.* 1. Let the line GA be drawn. Now, becauſe GD *a*
b 7. 3. = GA, and GB *b* ⊐ GA (ſince the right line FGB paſſes through F the center of the greater circle)
c 9. *ax.* therefore is GB ⊐ GD. *c Which is abſurd.*

PROP. XII.

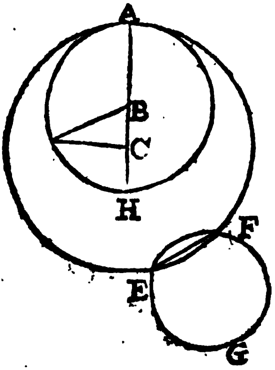


If two circles ACD, BCE touch one the other outwardly, the right line AB which joins their centers A, B, ſhall paſs thro' the point of contact C.

If it may be, let ADEB be a right line cutting the circles not in the point of contact C, but in the points D, E; draw AC, CB. then is
a 20. 1. AD + EB (AC + CB) *a* ⊏ ADEB. *b Which*
b 9. *ax.* *is abſurd.*

PROP.

PROP. XIII.

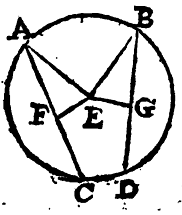


A circle CAF cannot touch a circle BAH in more points than one A, whether it be inwardly or outwardly.

1. Let one circle (if it can be) touch another in two
points A, H, a a 11. 3.
Then will the right line CB that joins the centers, if produced, fall as well in A, as H.
Now becauſe CH b=CA, and BH ⊏ CH, there- b 15. def. 1.
fore is BA (c BH) ⊏ CA. d *Which is abſurd.* c 15. def. 1.

2. If it be ſaid to touch outwardly in the points d 9. ax.
E and F, then draw the line EF, e which will e 2. 3.
be in both circles. Therefore thoſe circles cut one the other; Which is againſt the Hyp.

PROP. XIV.



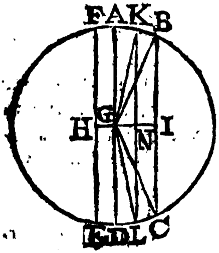
In a circle EABC equal right lines AC, BD are equally diſtant from the center E: and right lines AC, BD which are equally diſtant from the center, are equal among themſelves.

From the center E draw the perpendiculars EF, EG,
a which will biſect the lines AC,BD. join EA,EB. a 3. 3.
1 *Hyp.* AC=BD, therefore AF b=BG. But al- b 7. ax.

ſo $EA = EB$. therefore $EFq \; c = EAq - AFq =$
c 47.1.& $EBq - BGq \; c = EGq$. *d* Therefore $FE = EG$.
3. *ax.* 2.*Hyp.* $EF = EG$. Therefore $AFq \; c = EAq - EFq$
d *ſch*.48.1. $= EBq - EGq = BGq$. Therefore $AF \; d = GB$, and
e 6. *ax.* *e* conſequently $AC = BD$. *Which was to be dem.*

PROP. XV.

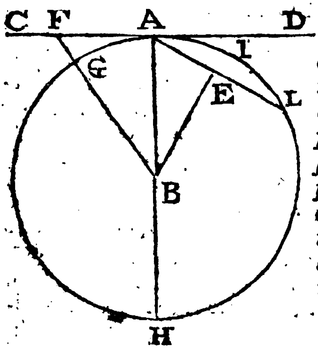
In a circle GABC the greateſt line is AD the diameter; and of all other lines, that line FE, which is neareſt to the center G is greater than any line BC farther diſtant from it.



1. Draw GB and GC.
a 15.*def*.1. The diameter AD (*a*
b 20, 1. $GB + GC$) *b* $\sqsubset BC$.

2. Let the diſtance GI be $\sqsubset GH$. Take $GN =$
GH. Thro' the point N draw KL perpendicularly
to GI: Join GK, GL. Becauſe $GK = GB$, and GL
c 24. 1. $= GC$, and the angle $KGL \sqsubset BGC$; *c* there-
fore is KL (FE) $\sqsubset BC$. *Which was to be dem.*

PROP. XVI.



A line CD drawn from the extreme point of the diameter HA of a circle BALH, perpendicular to the ſaid diameter, ſhall fall without the circle; and between the ſame right line and the circumference cannot be drawn another line AL. And the angle of *of the ſemicircle BAI, is greater than any right-lined acute angle BAL; and the remaining angle without the circumference DAI is leſs than any right-lined angle.*

1. From the center B to any point F in the right line AC, draw the right line BF. The ſide BF ſubtending the right angle BAF is *a* greater than the ſide BA which is oppoſite to the acute angle BFA. Therefore whereas BA (BG) reaches to the circumference, BF ſhall reach further; and ſo the point F. and for the ſame reaſon is any other point of the line AC placed without the circle.

a 19. 1.

2. Draw BE perpendicular to AL. The ſide BA oppoſite to the right angle BEA is *b* greater than the ſide BE which ſubtends the acute angle BAE; therefore the point E, and ſo the whole line EA falls within the circle.

b 19. 1.

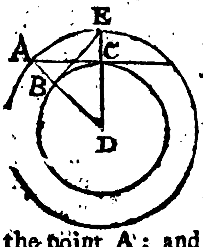
3. Hence it follows that any acute angle, to wit, EAD, is greater than the angle of contact DAI. and that any acute angle BAL is leſs than the angle of a ſemicircle BAI. *Which was to be dem.*

Coroll.

Hence, A right line drawn from the extremity of the diameter of a circle, and at right angles, is a tangent to the ſaid circle.

From this Propoſition are gathered many Paradox and wonderful Conſectaries, which you may meet with in the Interpreters.

PROP. XVII.

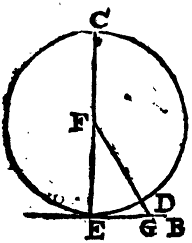


From a point given A to draw a right line AC which ſhall touch a circle given DBC.

From D the center of the circle given let a line DA cutting the circumference in B, be drawn to the point given A; from the center D deſcribe another circle thro' the point A; and from B draw a perpendicular to AD, which ſhall meet with the circle AE in the point E; and draw ED meeting with the circle BC in the point C. Then the line drawn from A to C ſhall touch the circle DBC.

a 15.*def*.1. For DB *a* = DC, and DE *a* = DA, and the
b 4. 1. angle D is common; *b* therefore the angle ACD
c *cor*.16.3. EBD and right. *c* Therefore AC touches the circle in C. *Which was to be done.*

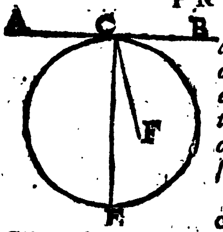
PROP. XVIII.



If any right line AB touch a circle FEDC, and from the center to the point of contact E a right line FE be drawn; that line FE ſhall be perpendicular to the tangent AB.

If you deny it, let ſome other line F G be drawn from the center F perpendicular to the tangent, and
a 2. *def*. 3. *a* cutting the circle in D. Therefore, whereas
& 16 3. the angle FGE is ſaid to be right, *b* thence is
b *cor*.17.1. the angle FEG acute; *c* ſo that FE (FD) ⊏
c 19. 1. FG. *Which is abſurd.*
d 9. *ax.*

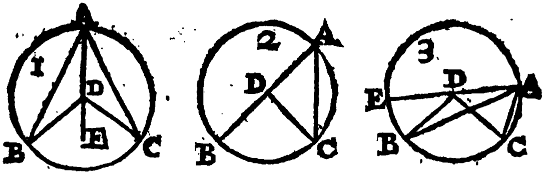
PROP. XIX.



If any right line AB touch a circle, and from the point of contact C a right line CE be erected at right angles to the tangent, the center of the circle ſhall be in the line CE ſo erected.

If you deny it; let the center be without the line CE in the point F; and from F to the point of
* 18. 3. contact let FC be drawn. Therefore the angle
a 11. *ax.* *FCB is right, and *a* conſequently equal to the
b 9. *ax.* angle ECB, which was right by Hyp. *b Which is abſurd.*

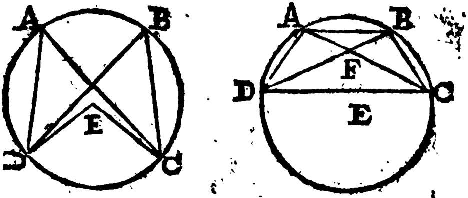
PROP. XX.



In a circle DABC, the angle BDC at the center is double of the angle BAC at the circumference, when the ſame arch of the circle BC is the baſe of the angles.

Draw the diameter ADE. The outward angle
BDE *a* = DAB + DBA *b* = 2 DAB. Likewiſe a 32. 1.
the angle EDC = 2 DAC. Therefore in the firſt b 5. 1.
caſe the whole angle *c* BDC = 2 BAC. and in c 2. *ax.*
the third caſe the *d* remaining angle BDC = 2 d 20. *ax.*
BAC. *Which was to be dem.*

PROP. XXI.

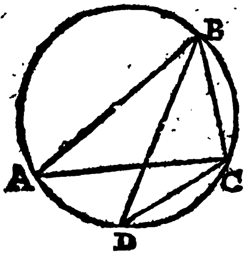


In a circle EDAC, the angles DAC and DBC which are in the ſame ſegment, are equal one to the other.

1. *Caſe.* If the ſegment DABC be greater than a
ſemicircle, from the center E draw ED, EC. Then
is twice the angle A *a* = E *a* = 2 B. *W. W. to be dem.* a 20. 3.
2. *Caſe.* If the ſegment be leſs than a ſemicircle,
then is the ſum of the angles of the triangle ADF
equal to the ſum of the angles of the triangle BCF.
from each let AFD be taken away *b* equal to b 15. 1.
BFC,

c *By the 1. caſe.* BFC, and ADB c=ACB be likewiſe taken away, then remains DAC=DBC. *W. W. to be dem.*

PROP. XXII.



The angles ADC, ABC of a quadrilateral figure ABCD deſcribed in a circle, which are oppoſite one to the other, are equal to two right angles.

a 32 [illegible]
b 21. 3.
c 1. ax.

Draw AC, BD. The angle ABC + BCA + BAC a = 2 right. But BDA b = BCA, and BDC b = BAC. c Therefore ABC + ADC = 2 right angles. *Which was to be dem.*

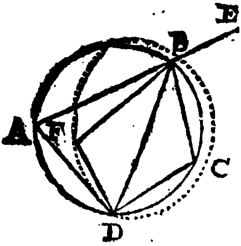
Coroll.

* *See the following Diagr.*

1. Hence, If one ſide * AB of a quadrilateral deſcribed in a circle be produced, the external angle EBC is equal to the internal angle ADC, which is oppoſite to that ABC which is adjacent to EBC, as appears by 13. 1. and 3. *ax.*

2. A circle cannot be deſcribed about a Rhombus; becauſe its oppoſite angles are greater or leſs than two right angles.

Schol.

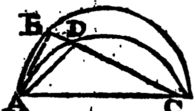


If in a quadrilateral ABCD the angles A and C, which are oppoſite, be equal to two right, then a circle may be deſcribed about that quadrilateral.

For a circle will paſs thro' any three angles (as ſhall appear by 5. 4.) I ſay that ſhall paſs thro' A the fourth alſo of ſuch

a quadrilateral: For if you deny it, let the circle paſs thro' F. Therefore the right lines BF, FD, BD being drawn, the angle $C + F$ *a* $= 2$ a 22. 3.
right *b* $= C + A$ wherefore, A *c* is equal to F. b *hyp.*
d Which is abſurd. c 3. *ax.*
d 21. 1.

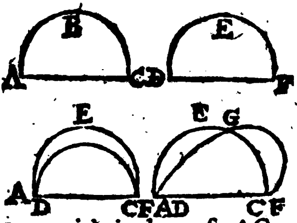
PROP. XXIII.



Two like and unequal ſegments of circles ABC, ADC cannot be ſet on the ſame right line AC, and the ſame ſide thereof.

For if they are ſaid to be like, draw the line CB cutting the circumference in D and B, join AB and AD. Becauſe the ſegments are ſuppoſed like; *a* therefore is the angle $ADC =$ a 10. *def.* 3.
ABC. *b Which is abſurd.* b 16. 1.

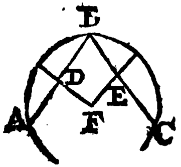
PROP. XXIV.



Like ſegments of circles ABC, DEF upon equal right lines AC, DF, are equal one to the other.

The baſe AC being laid on the baſe DF will agree with it, becauſe $AC = DF$. Therefore the ſegment ABC ſhall agree with the ſegment DEF (for otherwiſe it ſhall fall either within or without, and if ſo *a* then the ſegments are not like, a 23. 3.
which is contrary to the Hypotheſis, and at leaſt it ſhall fall partly within and partly without, and ſo cut in three points, *b* which is abſurd. *c* b 10. 3.
Therefore the ſegments $ABC = DEF$. *Which* c 8. *ax.*
was to be dem.

PROP. XXV.

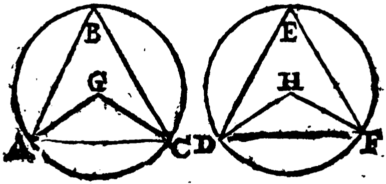


A ſegment of a circle ABC being given, to deſcribe the whole circle whereof that is a ſegment.

Let two right lines be drawn AB, BC, which biſect in the points D and E. From D and E draw the perpendiculars DF, EF meeting in the point F. I ſay this point ſhall be the center of the circle.

a cor. 1. 3. For the center ſhall be as well in *a* DF as ; FE, therefore it muſt be in the common point F. *Which was to be done.*

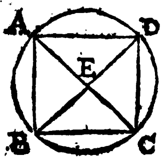
PROP. XXVI.



In equal circles GABC, HDEF, equal angles ſtand upon equal parts of the circumference, AC, DF; whether thoſe angles be made at the centers, G, H, or at the circumferences, B, E.

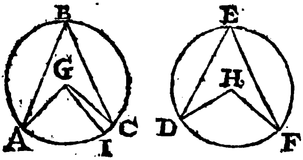
Becauſe the circles are equal, therefore is GA
= HD, and GC = HF; alſo by Hypotheſis
a 4. 1. the angle G = H; *a* therefore AC = DF.
b 20. 3. Moreover the angle B *b* = ½ G *c* = ½ H *b* = E.
c *hyp.* *d* Therefore the ſegments ABC, DEF are like,
d 10. *def.* 3. and *e* conſequently equal, *f* whence the remain-
e 24. 3. ing ſegments alſo AC, DF are equal. *Which*
f 3. *ax.* *was to be dem.*

Coroll.



In a circle ABCD let an arch AB be equal to DC; then ſhall AD be parallel to BC. For the right line AC being drawn, the angle ACB *a* = CAD; wherefore by 27. 1. the ſaid ſides are parallel. a 26. 3.

PROP. XXVII.

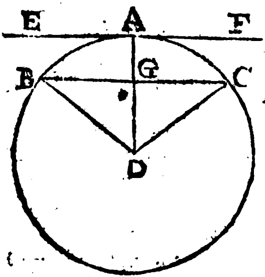


In equal circles GABC, HDEF, the angles ſtanding upon equal parts of the circumference AC, DF, are equal between themſelves, whether they be made at the centers G, H, or at the circumferences, B, E.

For if it be poſſible, let one of the angles AGC be ⊏ DHF, and make AGI = DHF. thence is the arch AI *a* = DF *b* = AC. *Which is abſurd.*

a 26. 3.
b *hyp.*
c 9. *ax.*

Schol.



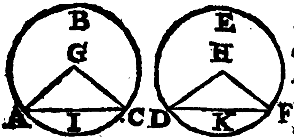
A right line EF, which being drawn from A. the middle point of any periphery BC, toucheth the circle, is parallel to the right line BC ſubtending the ſaid periphery.

From the center D draw a right line DA to the point of contact A, and join DB, DC.

The ſide DG is common, and DB = DC, and the angle BDA *a* = CDA, becauſe the arches BA, a 27. 3.

b *hyp.* CA are *b* equal) therefore the angles at the baſe
c 4. 1. DGB, DGC are *c* equal, and *d* conſequently
d 10. *def.* 1. right; But the inward angles GAE, GAF are
e *hyp.* alſo *e* right, *f* therefore BC, EF are parallel.
f 28. 1. *Which was to be dem.*

PROP. XXVIII.

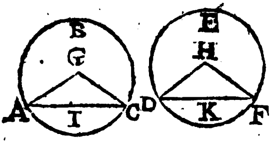


In equal circles GABC, HDEF, equal right lines AC, DF cut off equal parts of the circumference, the greateſt ABC equal to the greateſt DEF, and the leaſt AIC to the leaſt DKF.

From the centers G, H draw GA GC, & HD, HF
a *hyp.* Becauſe GA = HD, and GC = HF, and AC *a*
b 8. 1. = DF, *b* therefore is the angle G = H; *c* whence
c 26. 3. the arch AIC = DKF; *d* and ſo the remaining
d 3. *ax.* arch ABC = DEF. *Which was to be dem.*

But if the ſubtended line AC be ⊏ or ⊐ than DF, then in like manner will the arch AC be ⊏ or ⊐ than DF.

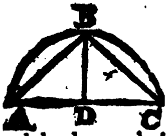
PROP. XXIX.



In equal circles GABC, HDEF, equal right lines AC, DF ſubtend equal peripheries ABC, DEF.

Draw the lines GA, GC, and HD, HF. Becauſe GA = HD, and GC = HF, and (becauſe the arches AC, DF
a *hyp.* are *a* equal) the angle G *b* = H. *c* therefore is
b 27. 3. the baſe AC = DF. *Which was to be dem.*
c 4. 1. This and the three precedent Propoſitions may be underſtood alſo of the ſame circle.

PROP. XXX.

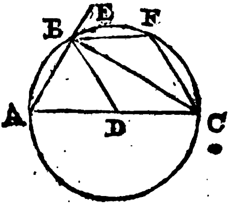


To cut a periphery given ABC into two equal parts.

Draw the right line AC, and biſect it in D; from D draw a perpendicular DB meeting with the arch in B, it ſhall biſect the ſame.

For join AB, and CB. The ſide DB is common, and AD *a* = DC, and the angle ADB *b* = a *conſtr.*
CDB. *c* therefore AB = BC; *d* whence the arch b 12. *ax.*
AB = BC. *Which was to be done.* c 4. 1.
d 28. 3.

PROP. XXXI.



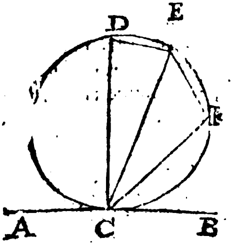
In a circle, the angle ABC, which is in the ſemicircle, is a right angle; but the angle, which is in the greater ſegment BAC is leſs than a right angle; and the angle which is in the leſſer ſegment BFC is greater than a right angle. Moreover, the angle of the greater ſegment is greater than a right angle, and the angle of the leſſer ſegment is leſs than a right angle.

From the center D draw DB, becauſe DB =
DA, therefore is the angle A *a* = DBA, and the a 5. 1.
angle DCB *a* = DBC, *b* therefore the angle ADC b 2. *ax.*
= A + ACB *c* = EBC, *d* ſo that ABC and EBC c 32. 1.
are right angles. *W. W. to be dem. e* Therefore d 10. *def.* 1.
BAC is an acute angle. *f W. W. to be dem.* And e *cor.* 17. 1.
further, whereas BAC *f* + BFC = 2 right, there- f 22. 3.
fore BFC is an obtuſe angle. Laſtly, the angle contained under the right line CB, and the arch BAC is greater than the right angle ABC; but the angle made by the right line CB and the periphery of the leſſer ſegment BFC *g* is leſs than g 9. *ax.*
the right angle ABC. *Which was to be dem.*

Schol.

In a right-angled triangle ABC, if the Hypothenuſe (or ſubtended line) AC be biſected in D, a circle drawn from the center D thro' the point A ſhall alſo paſs thro' the point B: As you may eaſily demonſtrate from this Prop. and 21. 1.

PROP. XXXII.

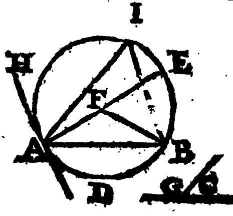


If a right line AB touch a circle, and from the point of contact be drawn a right line CE cutting the circle, the angles ECB, ECA which it makes with the tangent line are equal to thoſe angles EDC, EFC which are made in the alternate ſegments of the circle.

Let CD, the ſide of the angle EDC, be per-
a 26. 3. pendicular to AB (*a* for it's to the ſame purpoſe)
b 19. 3. *b* therefore CD is the diameter. *c* therefore the
c 31. 3. angle CED in a ſemicircle is a right angle, *d* and
d 32. 1. therefore the angle $D + DCE =$ to a right an-
e *conſtr.* gle $e = ECB + DCE$. *f* Therefore the angle D
f 3. *ax.* $= ECB$. *Which was to be dem.*

g 13. 1. Now whereas the angle $ECB + ECA$ $g =$ a
h 22. 3. right $h = D + F$, from both of theſe take away
k 3. *ax.* ECB and D, which are equal, *k* then remains
$ECA = F$. *Which was to be dem.*

PROP. XXXIII.

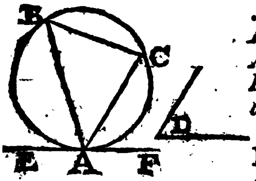


Upon a right line AB to deſcribe a ſegment of a circle AIEB which ſhall contain an angle AIB equal to a right lined angle given C.

a 23. 1. *a* Make the angle

BAD = C. Thro' the point A draw the line AE perpendicular to HD. At one end of the line given AB make an angle ABE = BAF; one ſide whereof let it cut the line AE in F; from the center F thro' the point A, deſcribe a circle which ſhall paſs thro' B (becauſe the angles FBA *b* = FAB. and *c* therefore FB = FA,) AIB b *conſtr.* is the ſegment ſought. For becauſe HD is per- c 6. 1. pendicular to the diameter AE, it therefore *d* d *cor.*16.3. touches the circle HD which AB cuts. And e 32. 3. therefore the angle AIB *e* = BAD *f* = C. *Which* f *conſtr.* *was to be done.*

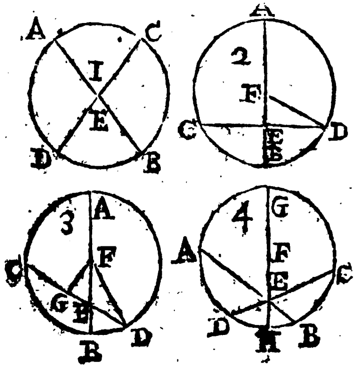
PROP. XXXIV.



From a circle given ABC to cut off a ſegment ABC containing an angle B equal to a right-lined angle given D.

a Draw a right line a 17. 3. EF which ſhall touch the b 23. 1. circle given in A, *b* let c 32. 1. AC be drawn alſo making an angle FAC = D. d *conſtr.* This line ſhall cut off ABC containing an angle B *c* = CAF *d* = D. *Which was to be done.*

PROP. XXXV.



If in a circle FBCA two right lines AB, DC cut each other, the rectangle comprehended under the ſegments AE, EB of the one, ſhall be equal to the

the rectangle comprehended under the ſegments **CE, ED** *of the other.*

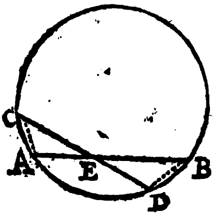
1. *Caſe.* If the right lines cut one the other in the center, the thing is evident.

2. *Caſe.* If one line AB paſs thro' the center F, and biſect the other line CD, then draw FD.
a 5. 2. Now the rectangle AEB+FEq *a* FBq *b*=FDq *c*
b *ſch.* 48. 1. = EDq + FEq *d* = CED+FEq. *e* Therefore the
c 47. 1. rectangle AEB=CED. *Which was to be dem.*
d *hyp.*
e 3. *ax.*

3. *Caſe.* If one of the lines AB be the diameter, and cut the other line CD unequally, biſect CD by FG a perpendicular from the center.

These are equal,
The rectangle AEB + FEq.
f 5. 2. *f* FBq (FDq.)
g 47. 1. *g* FGq + GDq.
h 5. 2. EGq + *h* GEq + Rectang. CED.
k 47. 1. *h* FEq + CED.
l 3. *ax.* *l* Therefore the Rectangle AEB = CED.

4. *Caſe.* If neither of the right lines AB, CD paſs thro' the center, then thro' the point of interſection E, draw the diameter GH. By that which hath been already demonſtrated, it appears that the rectangle AEB = GEH = CED. *Which was to be dem.*

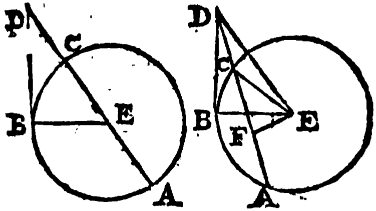


More eaſily, and generally, thus; join AC and BD, then becauſe the angles
a 15. 1. *a* CEA, DEB, and
b 21. 3. *b* alſo C, B (upon the ſame arch AD) are equal, thence
c *cor.* 31. 1. are the triangles CEA BED *c* equiangular. *d* Wherefore
d 4. 6. CE, EA :: EB, ED. and *e*
e 16. 6. conſequently CE × ED = AE × EB. *Which was to be dem.*

The citations out of the ſixth Book, both here and in the following Prop. have no dependance upon the ſame; ſo that it was free to uſe them.

PROP.

PROP. XXXVI.



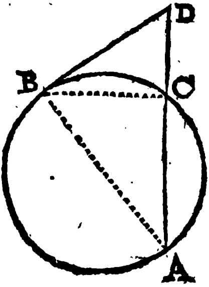
If any point D be taken without a circle E B C, and from that point two right lines D A, D B fall upon the circle, whereof one D A cut the circle, the other D B touches it, the rectangle comprehended under the whole line D A that cuts the circle, and under D C that part which is taken from the point given D to the convex of the periphery, ſhall be equal to the ſquare made of the tangent line.

1. *Caſe.* If the ſecant AD paſs thro' the cen-
ter, then join EB, this *a* will make a right an- a 18. 1.
gle with the line DB, wherefore DBq + *d* EBq b 47. 1.
(ECq) *b* = EDq *c* = AD × DC + ECq. There- c 6. 2.
fore AD × DC = DBq. *Which was to be dem.* d 3. *ax.*

2. *Caſe.* But if AD paſs not thro' the center
then draw EC, EB, ED, and EF perpendicular
to AD, *a* wherefore AC is biſected in F. a 3. 3.

Becauſe BDq + EBq *b* = DEq *b* = EFq + b 47. 1.
FDq *c* = EFq + ADC + FCq *d* = ADC + c 6. 2.
CEq (EBq.) *e* Therefore is BDq = ADC. *Which* d 47. 1.
was to be dem. e 3. *ax.*

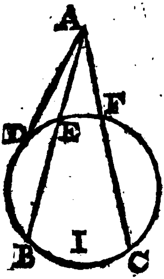
More



a 32. 3.
b 32. 1.
c 4. 6.
d 17. 6.

More eaſily, and generally, thus; Draw AB and BC. Then becauſe the angles A, and DBC *a* are equal, and the angle D common to both, thence are the triangles BDC, ADB *b* equiangular, *c* Wherefore AD. DB :: DB.CD. and *d* conſequently $AD \times DC = DBq$, *Which was to be dem.*

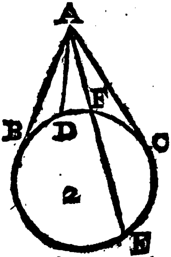
Coroll.



1. Hence, If from any point A taken without a circle, there be ſeveral lines AB, AC drawn which cut the circle, the rectangles comprehended under the whole lines AB, AC, and the outward parts AE, AF are equal between themſelves.

a 36. 3.

For if the tangent AD be drawn, then is $CAF = ADq$ *a* $= BAE$.



2. It appears alſo from hence, that if two lines AB, AC drawn from the ſame point do touch a circle, theſe two lines are equal one to the other.

a 36. 3.
b 36. 3.

For if AE be drawn cutting the circle, then is ABq *a* $=$ EAF *b* $= ACq$.

3. It is alſo evident that from a point A taken without a circle, there can be drawn but two lines AB, AC that ſhall touch the circle.

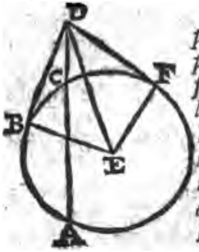
c 2. *cor.*
d 8. 3.

For if a third line AD be ſaid to touch the circle, thence is AD *c* = AB *c* = AC. *d Which is abſurd.*

4. And on the contrary, it is plain, that if two equal right lines AB, AC fall from any point A upon the convex periphery of a circle, and that if one of theſe equal lines AB touch the circle, then the other AC touches the circle alſo.

For if poſſible, let not AC, but another line
AD, touch the circle; therefore is AD *e* = AC e 2. cor.
f AB. *g Which is abſurd.* f hyp.
g 8. 3.

PROP. XXXVIII.



If without a circle EBF any point D be taken, and from that point two right lines DA, DB fall on the circle, whereof one line DA cuts the circle the other DB falls upon it; and if alſo the rectangle comprehended under the whole line that cuts the circle, and under that part of it DC which is taken betwixt the point D and the convex periphery, be equal to that ſquare which is made of the line DB falling on the circle, I ſay that the line DB ſo falling ſhall touch the circle given.

From that point D *a* let a tangent DF be a 17. 3.
drawn, and from the center E draw ED, EB, EF.
Now becauſe DBq *b* = ADC *c* = DFq, there- b hyp.
fore is DB *d* = DF: But EB = EF, and the ſide c 36. 3.
ED common; *e* therefore the angle EBD = EFD. d 1. ax. &
but EFD is a right angle, and *f* therefore EBD ſch. 4. 1.
is right alſo, and *g* therefore DB touches the e 8. 1.
circle. *Which was to be dem.* g cor. 16. 3.

Coroll. h 8. 1.

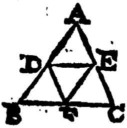
From hence it follows that the *h* angle EDB = EDF.

The End of the third Book.

THE FOURTH BOOK OF EUCLIDE's ELEMENTS.

Definitions.

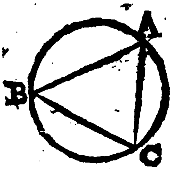
I. A Right-lined figure is ſaid to be inſcribed in a right-lined figure, when every one of the angles of the inſcribed figure touch every one of the ſides of the figure wherein it is inſcribed.

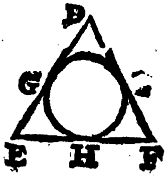


So the triangle DEF is inſcribed in the triangle ABC.

II. In like manner a figure is ſaid to be deſcribed about a figure, when every one of the ſides of the figure circumſcribed touch every one of the angles of the figure about which it is circumſcribed.

So the triangle ABC is deſcribed about the triangle DEF.



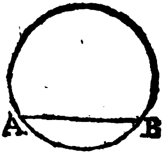


III. A right-lined figure is ſaid to be inſcribed in a circle, when all the angles of that figure which is inſcribed do touch the circumference of the circle.

IV. A right-lined figure is ſaid to be deſcribed about a circle, when all the ſides of the figure which is circumſcribed touch the periphery of the circle.

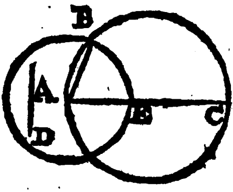
V. After the like manner a circle is ſaid to be inſcribed in a right-lined figure, when the periphery of the circle touches all the ſides of the figure in which it is inſcribed.

VI. A circle is ſaid to be deſcribed about a figure, when the periphery of the circle touches all the angles of the figure, which it circumſcribes.



VII. A right line is ſaid to be co-apted or applied in a circle when the extremes thereof fall upon the circumference; *as the right line AB.*

PROP. I. *Probl.* 1.

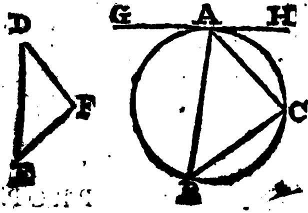


In a circle given ABC to apply a right line AB equal to a right line given D, which doth not exceed AC the diameter of the circle.

From the center A by the ſpace AE = D a deſcribe a circle meeting with the circle given in B, draw AB. Then is AB b = AE c = D. *Which was to be done.*

a 3. *poſt.* & 3. 1.
b 15. *def.* 1.
c *conſtr.*

PROP. II. *Probl.* 2.



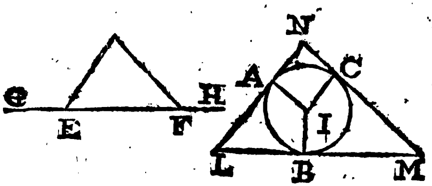
In a circle given ABC to deſcribe a triangle ABC, equiangular to a triangle given DEF.

Let the right line

a 17. 3. line GH *a* touch the circle given in A; *b* make
b 23. 1. the angle HAC = E, *b* and the angle GAB =
F. then join BC; and the thing is done.

c 32. 3. For the angle B *c* = HAC *d* = E, and the
d *conſtr.* angle C *c* = GAB *d* = F; *e* whence alſo the
e 32. 1. angle BAC = D. Therefore the triangle BAC
inſcribed in the circle is equiangular to DEF. *Which was to be done.*

PROP. III.



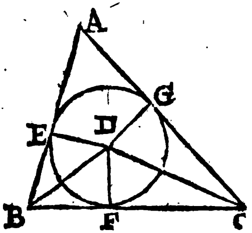
About a circle given IABC to deſcribe a triangle LNM equiangular to a triangle given DEF.

Produce the ſide EF on both ſides; at the
a 23. 1. center I *a* make an angle AIB = DEG, and an
angle BIC = DFH. Then in the points A, B, C
b 17. 3. let three right lines LN, LM, NM *b* touch
the circle, and the thing is done.

For it's evident that the right lines LN, LM,
c 13. 1. MN will meet and make a triangle, *c* becauſe
d 11. 3. the angles LAI, LBI are right; ſo that the *d*
right line AB produced will make the angles
LAB, LBA, leſs than two right angles.

a *ſch.* 32. 1. Since, therefore the angle AIB + L *e* = 2
f 13. 1. right angles *f* = DEG + DEF, and AIB *g* =
g *conſtr.* DEG; *h* therefore is the angle L = DEF. By
h 3. *ax.* the like way of argument the angle M = DEF.
k 32. 1. *k* therefore alſo the angle N = D. And therefore the triangle LNM deſcribed about the circle is equiangular to EDF the triangle given. *Which was to be done.*

PROP. IV.



In a triangle given ABC, to deſcribe a circle EFG.

a Biſect the angles B and C with the right lines BD, CD meeting in the point D, *b* and draw the perpendiculars D E, DF, DG. A circle deſcribed from the center D thro' E, will paſs thro' G and F, and touch the three ſides of the triangle. a 9. 1. b 12. 1.

For the angle BDE *c* = DBF ; and the angle DEB *d* = DFB ; and the ſide DB common. *e* therefore DE = DF. By the like argument DG = DF. The circle therefore deſcribed from the center D paſſes thro' the three points E, F, G. and whereas the angles at E, F, G are right, therefore it touches all the ſides of the triangle. *Which was to be done.* c *conſtr.* d 12. *ax.* e 26. 1.

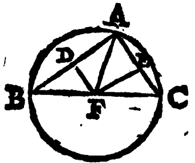
Schol.

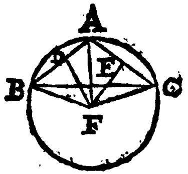
Hence, *The ſides of a triangle being known, their ſegments which are made by the touchings of the circle inſcribed ſhall be found*, thus ; *Pet. Herig.*

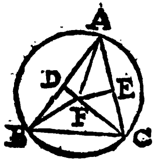
Let AB be 12, AC 18, BC 16, then is AB + BC = 28. Out of which ſubduct 18 = AC = AE + FC, then remains 10 = BE + BF. Therefore BE, or BF = 5 ; and conſequently FC, or CG, = 11. Wherefore GA, or AE, = 7.

PROP.

PROP. V.







About a triangle given ABC to deſcribe a circle FABC.

a 10. *and* 11. 1. *a* Biſect any two ſides BA, CA with a perpendicular DF, EF meeting in the point F. I ſay this ſhall be the center of the circle.

For, let the right lines FA, FB, FC be drawn.
b *conſtr.* Now becauſe AD *b* = DB and the ſide DF com-
c *conſtr.* & mon, and the angles FDA *c* = FDB, therefore is
11. *ax.* FB *d* = FA. After the ſame manner is FC =
d 4. 1. FA. Therefore a circle deſcribed from the center F ſhall paſs thro' the angles of the triangle given (*viz.*) B, A, C. *Which was to be done.*

Coroll.

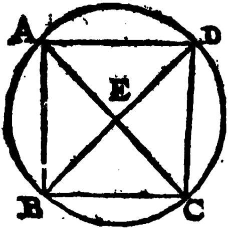
* 31. 3. * Hence, if a triangle be acute-angled, the center ſhall fall within the triangle; if right-angled, in the ſide oppoſite to the right angle, and if obtuſe-angled, without the triangle.

Schol.

By the ſame method may a circle be deſcribed, that ſhall paſs thro' three points given, not being in the ſame ſtrait line.

PROP.

PROP. VI.

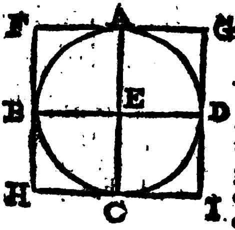


In a circle given EABCD to inſcribe a ſquare ABCD.

a Draw the diameters AC, BD cutting each other at right angles in the center E. Join the extremes of theſe diameters with the right lines AB, BC, CD, DA. *And the thing is done.* a 11. 1.

Now becauſe the four angles at E are right, the *b* arches and *c* ſubtended lines AB, BC, CD, DA b 26. 3.
are equal; therefore is the figure ABCD equi- c 29. 3.
lateral, and all the angles in ſemicircles, and ſo d 31. 3.
d right. *e* Therefore ABCD is a ſquare inſcri- e 29. 3. def.
bed in a circle given. *Which was to be done.* 1.

PROP. VII.

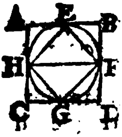


About a circle given EABCD to deſcribe a ſquare FHIG.

Draw the diameters AC, BD cutting one the other at right angles; thro' the extremes of theſe diameters *a* draw tangents meeting in F, H, I, G, then I ſay it's done. a 17. 3.

For becauſe *b* the angles A and C are right, *c* b 18. 3.
therefore is FG parallel to HI. After the ſame c 18. 1.
manner is FH parallel to GI. and therefore FHIG is a Pgr, and alſo right-angled. It is equilateral
becauſe FG *d* = HI *d* = DB *e* = CA *d* = FH *d* = d 34. 1.
GI. Wherefore FHIG is a *f* ſquare deſcribed e 15. def. 1.
about the circle given. *Which was to be done.* f 29. def. 1.

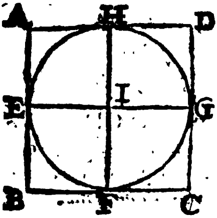
Schol.



A ſquare ABCD deſcribed about a circle is double of the ſquare E F G H inſcribed in the ſame circle.

For the rectangle H B = 2 △HEF, and HD = 2 HGF by the 41. 1.

PROP. VIII.

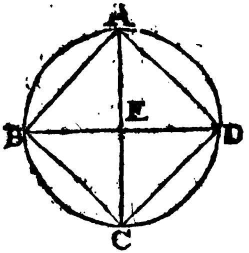


In a ſquare given ABCD to inſcribe a circle IEFGH.

Biſect the ſides of the ſquare in the points H, E, F, G, cutting one the other in I. A circle drawn from the center I through H ſhall be inſcribed in the ſquare.

a 7. *ax.* For becauſe AH and BF are *a* equal and *b* pa-
b *hyp.* rallel, *c* therefore is AB parallel to HF parallel
c 33. 1. to DC. After the ſame manner is AD parallel
to EG, parallel to BC; therefore IA, ID, IB,
d 7. *ax.* IC are parallelograms. Therefore AH *d* = AE
e 34. 1. *e* = HI = EI = FI = IG. The circle therefore deſcribed from the center I thro' H ſhall paſs thro' H, E, F, G, and touch the ſides of the ſquare, being the angles H, E, F, G, are right. *Which was to be done.*

PROP. IX.

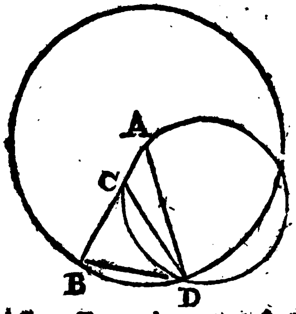


About a ſquare given ABCD to deſcribe a circle EABCD.

Draw the diameters AC, BD cutting one the other in E. From the center E thro' A deſcribe a circle, then I ſay that circle is deſcribed about the ſquare.

For the angles ABD and BAC are *a* half of a 4. Cor.
right angles, *b* therefore EA = EB. After the 33. 1.
ſame manner is EA = ED = EC. The circle b 6. 4.
therefore deſcribed from the center E paſſes
thro' A, B, C, D the angles of the ſquare given.
Which was to be done.

PROP. X.



To make an Iſoſceles triangle A B D, having each angle at the baſe B and ADB double to the remaining angle A.

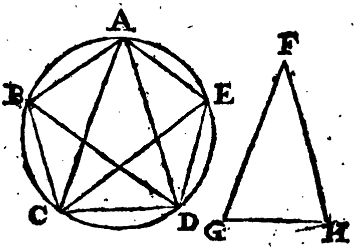
Take any right
line AB, and di-
vide it in C, *a* ſo a 11. 2.
that $AB \times BC$
may be equal to
ACq. From the center A thro' B, deſcribe the
circle ABD; and in this circle *b* apply BD = b 1. 4.
AC, and join AD; I ſay ABD is the triangle
required.

For, draw DC, and thro' the points C, D, A.
c 5. 4. *c* draw a circle. Now becauſe AB×BC = ACq, *d*
d 37. 3. it is evident that FD touches the circle ACD
e 32. 3. which CD cutteth; *e* therefore is the angle BDC
f 2. *ax.* = A, and therefore the angle BDC + CDA *f* =
g 32. 1. A + CDA *g* = BCD. But BDC + CDA = BDA
h 5. 1. *h* = CBD. *k* therefore the angle BCD = CBD,
k 1. *ax.* and therefore DC *l* = DB = *m* AC, *n* wherefore
l 6. 1. the angle CDA = A = BDC. therefore ADB
m *conſtr.* = 2 A = ABD. *Which was to be done.*
n 5. 1. This conſtruction is Analytically found out
thus; take the thing for done, and let the right
line DC biſect the angle BDA; *a* therefore DA,
a 3. 6. DB :: CA, CB. alſo becauſe the angle CDA *b* =
b *conſtr.* $\frac{1}{2}$ ADB *c* = A, *c* therefore CA = DC. and be-
c *hyp.* cauſe the angle DCB *b* = A CDA = 2 A *e* =
d 6. 1. B, *d* thence will be DB = DC. *f* from whence
e 31. 1. alſo DB = CA. and ſo DA(BA.)CA :: CA.CB. *g*
f 2. *ax.* whence BA × CB = CAq.
g 17. 6.

Coroll.

Whereas all the angles A, B, D *h* make up
h 32. 1. two right angles, it's evident that A is $\frac{1}{5}$ of two
right angles.

PROP. XI.



In a circle given ABCDE to deſcribe a Pentagone figure ABCDE equilateral and equiangular.

a 10. 4. *a* Deſcribe an Iſoſceles triangle FGH, having
b 2. 4. each angle at the baſe double to the other; *b* in-

ſcribe

ſcribe a triangle CAD equiangular to the ſaid triangle FGH. *c* Biſect the angles at the baſe c 9. 1.
ACD and ADC with the right lines DB, CE meeting with the circumference in B and E. join the right line CB, BA, AE, ED. *Then I ſay it is done.*

For it is evident by conſtruction that the angles CAD, CDB, BDA, DCE, ECA, are equal; wherefore the *d* arches and *e* ſubtended lines DC, d 26. 3.
CB, BA, AE, DE are equal. Therefore the Pen- e 29. 3.
tagone is equilateral, and equiangular *f* becauſe f 27. 3.
the angles of it BAE, AED, *&c.* ſtand on equal *g* arches BCDE, ABCD, *&c.* g 2. ax.

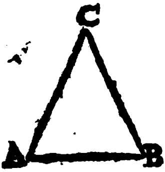
A more eaſy Practice of this Problem ſhall be deliver'd at 10, 13.

Coroll.

Hence, each angle of an equilateral and equiangular Pentagone is equal to $\frac{3}{5}$ of two right angles, or $\frac{6}{5}$ of one right angle.

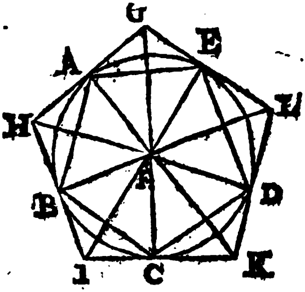
Schol.

Generally all figures of odd number of ſides are Pet.Herig.
inſcribed in circles by the help of Iſoſceles triangles, whoſe angles at the baſe are multiples of thoſe at the top: and figures of even number of ſides are inſcribed in a circle by the help of Iſoſceles triangle, whoſe angles at the baſe are multiples ſeſquialter of thoſe at the top.



As in the Iſoſceles triangle CAB, if the angle $A = 3C = B$, then will AB be the ſide of a Heptagone. If $A = 4C$, then is AB the ſide of Enneagone. But if $A = \frac{1}{2}C$, then is AB the ſide of a ſquare. And if $A = 2\frac{1}{2}C$ CAB will ſubtend the ſixth part of a circumference: and likewiſe if $A = 3\frac{1}{2}$, then will AB be the ſide of Octagone.

PROP. XII.



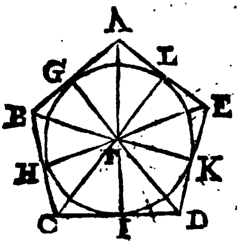
About a circle given FABCDE, to deſcribe an equilateral and an equiangular pentagone HIKLG.

a 11. 4. *a* Inſcribe a pentagone ABCDE in the circle
given; and from the center draw the right lines
FA, FB, FC, FD, FE; and to thoſe lines draw ſo
many perpendiculars GAH, HBI, ICK, KDL, LEG
meeting in the points H, I, K, L, G. then I ſay it
is done. For becauſe GA, GE from the ſame
b cor. 16. 3. point G *b* touch the circle, *c* therefore is GA =
GE, and *d* therefore the angle GFA = GFE,
c 2. cor. 36. 3. therefore the angle AFE = 2 GFA. After the
ſame manner is the angle AFH = FHB, and
d 8. 1. conſequently the angle AFB = 2 AFH. *e* But
e 27. 3. the angle AFE = AFB, *f* therefore the angle GFA
f 7. ax. = AFH. But alſo the angle FAH *g* = FAG,
g 12. ax. and the ſide FA is common, *h* therefore HA =
h 26. 1. AG = GE = EL, &c. *k* Therefore HG, GL, LK,
k 3. ax. KI, IH the ſides of the pentagone are equal, the
angles alſo are, becauſe double of the equal angles AGF, AHF, therefore, &c.

Coroll.

After the ſame manner, if any equilateral and equiangled figure be deſcribed in a circle, and at the extreme points of the ſemi-diameters drawn from the center to the angles, be drawn perpendicular lines to the ſaid diameters, I ſay that theſe perpendiculars ſhall make another figure of as many equal ſides and equal angles, deſcribed about the circle.

PROP. XIII.



In an equilateral and equiangular pentagone given ABCDE, to inſcribe a circle FGHK.

a Biſect two angles a 9. 1.
of the pentagone A
and B with the right
lines AF, BF meeting
in the point F. From
F draw the perpendiculars FG, FH, FI, FK, FL.
Then a circle deſcribed from the center F thro'
G will touch all the ſides of the pentagone.

Draw FC, FD, FE. Becauſe BA *b* = BC, and b *hyp.*
the ſide BF common, and the angle FBA *c* = FBC, c *conſtr.*
d therefore is AF = FC and the angle FAB = FCB, d 4. 1.
but the angle FAB *e* = ½ BAE = ½ BCD. There- e *hyp.*
fore the angle FCB = ½ BCD. After the ſame
manner are all the whole angles C, D, E biſected.
Now whereas the angle FGB *f* = FHB, and the f 12. *ax.*
angles FBH = FBG and the ſide FB is common, *g* g 26. 1.
therefore is FG = FH. In like manner are all the
right lines FH, FI, FK, FL, FG equal. Therefore
a circle deſcribed from the center F thro' G paſſes
thro' the points H, I, K, L and *h* touches the ſides h *cor.* 16. 3.
of the pentagone, becauſe the angles at thoſe
points are right. *Which was to be done.*

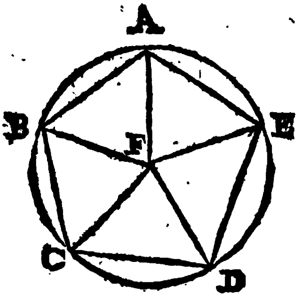
Coroll.

Hence, If any two neareſt angles of an equilateral and equiangular figure be biſected, and from that point in which the lines meet that biſect the angles be drawn right lines to the remaining angles of the figure, all the angles of the figure ſhall be biſected.

Schol.

By the ſame method ſhall a circle be inſcribed in any equilateral or equiangular figure.

PROP. XIV.



About a pentagone given ABCDE equilateral and equiangular to deſcribe a circle FABCDE.

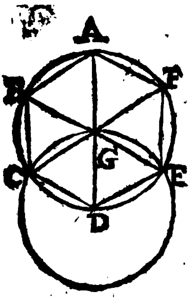
Biſect any two angles of the pentagone with the right lines AF, BF meeting in the point F; the circle deſcribed from the center F thro' A ſhall be deſcribed about the pentagone.

a cor. 13. 4. For let FC, FD, FE be drawn. *a* Then the an-
b 6. 1. gles C, D, E are biſected; *b* and therefore FA,
FB, FC, FD, FE are equal; therefore the circle deſcribed from the center F paſſes thro' A, B, C, D, E all the angles of the pentagone. *Which was to be dem.*

Schol.

By the ſame art is a circle deſcribed about any figure which is equilateral and equiangular.

PROP. XV.



In a circle given GABCDEF to inſcribe an Hexagone (or ſix-ſided figure) equilateral and equiangular ABC DEF.

Draw the diameter AD; from the center D through the center G deſcribe a circle cutting the circle given in the points C and E. Draw the diameters CF, EB; and join AB, BC, CD, DE, EF, FA. *Then I ſay it's done.*

For the angle CGD [a] = $\frac{1}{3}$ of 2 right [a] = DGE [b] = AGF [b] = AGB. [c] Therefore BGC = $\frac{1}{3}$ of 2 right FGE; therefore the [d] arches and [e] ſubtenſes AB, BC, CD, DE, EF are equal. Therefore the Hexagone is equilateral, but it is equiangled alſo, [f] becauſe all the angles of it ſtand upon equal arches.

a 32. 1.
b 15. 1.
c *cor.* 13. 1.
d 26. 3.
e 29. 3.
f 27. 3.

Coroll.

1. Hence, The ſide of an Hexagone inſcribed in a circle is equal to the ſemidiameter.

2. Hereby an equilateral triangle ACE may very eaſily be deſcribed in a circle given.

Schol.

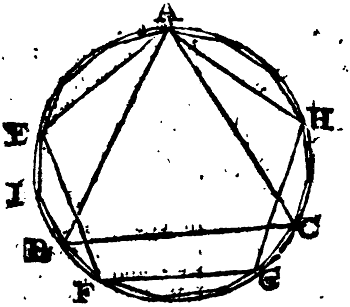
To make a true Hexagone upon a right line given CD.

[a] Make an equilateral triangle CGD upon the line given CD; from the center G thro' C and D deſcribe a circle. That circle ſhall contain the Hexagone made upon the given line CD.

Andr. Tacq.
a 1. 1.

PROP.

PROP. XVI.



In a circle given AEBC to inſcribe a quindecagone (or fifteen-ſided figure) equilateral and equiangular.

a 11. 4. *a* Inſcribe an equilateral pentagone AEFGH
b 3. 4. in the circle given, and *b* alſo an equilateral triangle ABC. then I ſay BF is the ſide of the quindecagone required.

c conſtr. For the arch AB *c* is $\frac{1}{3}$ or $\frac{5}{15}$ of that periphery whereof AF is $\frac{2}{5}$ or $\frac{6}{15}$. therefore the remaining part BF is $\frac{6}{15}$ of the periphery; and therefore the quindecagone, whoſe ſide is BF, is equilateral;
d 27. 3. but it is equiangular alſo, *d* becauſe all the angles inſiſt on equal arches of a circle, where every one $\frac{13}{15}$ of the whole circumference. Therefore, &c.

Schol.

A circle is geometrically divided into parts.
{ 4, 8, 16, &c. by 6, 4, and 9, 1.
3, 6, 12, &c. by 15, 4, and 9, 1.
5, 10, 20, &c. by 11, 4, and 9, 1.
15, 30, 60, &c. by 16, 4, and 9, 1.

Any other way of dividing the circumference into any parts given, is as yet unknown, wherefore in the conſtruction of ordinate figures, we are forced to have recourſe to mechanick artifices, concerning which you may conſult the Writers of practical Geometry.

THE

THE FIFTH BOOK OF EUCLIDE's ELEMENTS.

Definitions.

I. A Part, is a magnitude of a magnitude, a leſs of a greater, when the leſs meaſures the greater.

II. Multiple is a greater magnitude in reſpect of a leſſer, when the leſſer meaſures the greater.

III. Ratio (or rate) is the mutual habitude or reſpect of two magnitudes of the ſame kind each to other, according to quantity.

In every ratio, that quantity which is referred to another quantity is called the antecedent of the ratio, and that to which the other is referred is called the conſequent of the ratio, as in the ratio of 6 *to* 4, 6 *is the antecedent, and* 4 *the conſequent.*

Note, *The quantity of any ratio is known by dividing the antecedent by the conſequent; as the ratio of* 12 *to* 5 *is expreſſed by* $\frac{12}{5}$, *or the quantity of the ratio of* A *to* B *is* $\frac{A}{B}$. *Wherefore, often for brevity ſake, we denote the quantities of rations thus;* $\frac{A}{B} \sqsubset$, or $=$, or $\sqsupset \frac{C}{D}$. *that is, the ratio of A to B is greater, equal, or leſs than the ratio of C to D. And this note muſt be diligently obſerved in the underſtanding of the following Book.*

Concerning the divers ſpecies of ratio's, you may pleaſe to conſult Interpreters.

IV. Proportion is a ſimilitude of ratio's.

That which is here termed Proportion, *is more rightly called* Proportionality *or* Analogy; *for* *proportion commonly denotes no more than the ratio betwixt two magnitudes.*

V. Thoſe numbers are ſaid to have a ratio betwixt them, which being multiplied may exceed one the other.

E,12	A, 5. B,6.	G,24.
F,30.	C,10.D,15.	H,60.

VI. Magnitudes are ſaid to be in the ſame ratio, the firſt A to the ſecond C, and the third C to the fourth D, when the equimultiples E and F of the firſt A, and the third C compared with the equimultiples G, H of the ſecond B and the fourth D, according to any multiplication whatſoever, either both together E, F are leſs than G, H both together, or equal taken together, or exceed one the other together, if thoſe be taken E, G and F, H, which anſwer one to the other.

The note hereof is :: ; as A.B :: C.D. That is, as A is to B, ſo is C to D. which ſignifies that A to B, and C to D are in the ſame ratio. We ſometimes thus expreſs it; $\frac{A}{C} = \frac{C}{D}$ *that is A.B :: C. D.*

VII. Magnitudes that have the ſame ratio (A. B :: C. D. are called proportional.

E,30.	A, 6. B,4.	G,28.
F,60.	C, 12. D,9.	H, 63.

VIII. When of equimultiples, E the multiple of the firſt magnitude A exceeds G the multiple of the ſecond B, but F the multiple of the third C exceeds not H the multiple of the fourth D, then the firſt A to the ſecond B has a greater ratio than the third C to the fourth D.

If $\frac{A}{B} \sqsubset \frac{C}{D}$, *it is not neceſſary from this definition that E ſhould always exceed G, when F is leſs than H; but it is granted that this may be.*

IX. Proportionality conſiſts in three terms at leaſt. *Whereof the ſecond ſupplies the place of two.*

X. When 3 magnitudes A, B, C are proportional, the firſt A ſhall have a duplicate ratio to the third C of that it has to the ſecond B: But when 4 magnitudes A, B, C, D are proportional, the firſt A ſhall have a triplicate ratio to the fourth D of what it had to the ſecond B; and ſo always in order one more, as the proportion ſhall be extended.

Duplicate ratio is thus expreſſed $\frac{A}{C} = \frac{A}{B}$ *twice, that is, the ratio of A to C is double of the ratio of A to B.*

Triple ratio is thus expreſſed; $\frac{A}{D} = \frac{A}{B}$ *thrice. That is the ratio of A to D is triple of the ratio of A to B.*

∺ *denotes continual proportionals;* as A, B, C, D; *or* 2, 6, 18, 64, *are* ∺.

XI. Magnitudes of a like ratio, are antecedents to antecedents, and conſequents to conſequents; *As if A.B :: C.D. A and C; and B and D are homologous or magnitudes of a like ratio.*

XII. Alternate proportion is the comparing of antecedent to antecedent, and conſequent to conſequent. *As if A.B :: C.D. therefore alternately, or by permutation, A.C :: B.D. by the* 16. *of* 5.

In this definition, and the 5 *following, names are given to the ſix ways of arguing which are often uſed by Mathematicians: the force of which inferences depends on the Propoſitions of this Book; which are named in their Explications.*

XIII. Inverſe ratio is when the conſequent is taken as the antecedent, and ſo compared to the antecedent as the conſequent; *As A. B* :: C. *D. therefore generally B. A* :: D. C. *by cor.* 4. 5.

XIV. Compounded ratio is when the antecedent and conſequent taken both as one are compared to the conſequent it ſelf. *As A. B :: C. D. therefore by compoſition A + B. B :: C + D. D by* 18. 5.

XV. Divided ratio is when the exceſs wherein the antecedent exceeds the conſequent, is compared to the conſequent. *As A. B :: C. D. therefore by diviſion A—B. B* :: *C—D. D. by* 17. 5.

XVI. Converſe ratio is when the antecedent is compared to the exceſs wherein the antecedent exceeds the conſequent. *As A. B :: C. D. therefore by converſe ratio A. A B :: C. C D. by the coroll. of the 19. of the 5.*

XVII. Proportion of equality is where there are taken more magnitudes than two in one order, and alſo as many magnitudes in another order, comparing two to two being in the ſame ratio; it comes to paſs that as in the firſt order of magnitudes, the firſt is to the laſt, ſo in the ſecond order of magnitudes is the firſt to the laſt. Or otherwiſe: it is a compariſon of the extremes together, the mean magnitudes being taken away.

XVIII. Ordinate proportionality is, when as the antecedent is to the conſequent, ſo is the antecedent to the conſequent, and as the conſequent is to any other, ſo is the conſequent to any other. *As A. B :: D. E. alſo B. C :: E. F. it ſhall be true alſo A. C :: D. F. by the 22. of the 5.*

XIX. Inordinate proportion is, when three magnitudes being put, and others alſo, which are equal to theſe in multitude, as in the firſt magnitudes the antecedent is to the conſequent, ſo in the ſecond magnitude is the antecedent to the conſequent; and as in the firſt magnitudes the conſequent is to any other, ſo in the ſecond magnitudes any other thing to the antecedent. *As A. B :: F. G. alſo B. C :: E. F. it ſhall be true in inordinate proportion. A. C :: E. G. by the 23. of the 5.*

XX. Any number of magnitudes being put; the proportion of the firſt to the laſt is compounded out of the proportions of the firſt to the ſecond, the ſecond to the third, and the third to the fourth, and ſo forwards till the proportion ariſe.

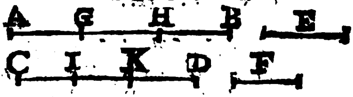
Let there be any number of magnitudes A, B, C, D. by this definition $\frac{A}{D} = \frac{A}{B} + \frac{B}{C} + \frac{C}{D}$.

Axiome.

Axiome.

Magnitudes equimultiples to the ſame multiple, are alſo equimultiples betwixt themſelves.

PROP. I.

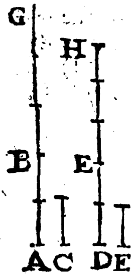


If there be a number of magnitudes how many ſoever, AB, CD equimultiples to a like number of magnitudes E, F each to other; how multiple one magnitude AB is one E, ſo multiples are all the magnitudes AB + CD to all the other magnitudes E + F.

Let AG, GH, HB the parts of the quantity AB, be equal to E, and alſo let CI, IK, KD the parts of the quantity CD be equal to F. The number of theſe are put equal to thoſe. Now whereas AG + CI *a* = E + F; *a* and GH + IK = E + F; *a* and HB + KD = E + F; it is evident that AB + CD does ſo often contain E + F as one AB contains E. *Which was to be done.*

a 2. ax.

PROP. II.



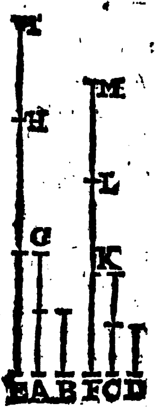
If the firſt AB be equimultiple to the ſecond C, as the third DE is to the fourth F, and if the fifth BG be equimultiple to the ſecond C as the ſixth EH is to the fourth F; then ſhall the firſt compounded with the fifth (AG) be equimultiple to the ſecond C, as the third compounded with the ſixth (DH) is to the fourth F.

The number of parts in AB equal each to C is put equal to the number of parts in DE. whereof each part is equal to F. Likewiſe the number of parts in BG is equal to the number of parts in EH. Therefore the number of parts in AB + BG is equal to the number of parts in DE + EH.

a 2. ax. FH. *a* That is, the whole line AG is as equimultiple of C, as the whole line DH is of F. *Which was to be dem.*

PROP. III.

If the firſt A be equimultiple of the ſecond B, and the third C of the fourth D, and there be taken EI, FM equimultiples of the firſt and third, then will each of the magnitudes taken be alike equimultiple of both, the one EI to the ſecond B, the other FM to the fourth D.



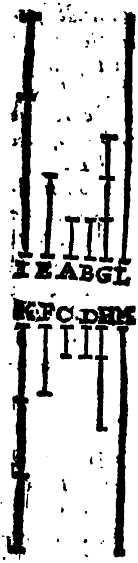
a hyp. Let EG, GH, HI the parts of the multiple EI be equal to A, alſo let FK, KL, LM the parts of the multiple FM be equal to F, *a* the number of theſe is equal to the number of thoſe. Moreover A (that is) EG or GH, or HI is put as equimultiple of B, as C, or FK, *&c.* of D. *b* Therefore EG + GH is equimultiple of the ſecond B, as FK + KL is of the fourth D. *c* By the ſame way of argument is EI (EH + HI) as multiple of B, as FM (FL + LM) is of D. *Which was to be done.*

b 2 5.

c 2. 5,

PROP.

PROP. IV.



If the firſt A have the ſame ratio to the ſecond B, as the third C to the fourth D; then alſo E and F the equimultiples of the firſt A and the third C, ſhall have the ſame ratio to G and H the equimultiples of the ſecond B and the fourth D, according to any multiplication, if ſo taken as they anſwer each to other. (E. G :: F. H.)

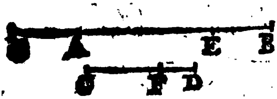
Take I and K the equimultiples of E and F; and alſo L and M the equimultiples of G and H. *a* Then is I as multiple of A, as a 3. 5.
K of C; *a* and alſo L is as multiple of B, as M of D. Therefore b *hyp.*
whereas it is A. B :: C. D; according to the ſixth definition, if I be ⊏, =, ⊐ L, then conſequently after the ſame manner is K ⊏, =, ⊐, M. Therefore when I and K are taken as multiples of E and F, as L and M of G, and H, then will it be by the ſeventh definition E. G :: F. H. *Which was to be dem.*

Coroll.

From hence is wont to be demonſtrated the proof of inverſe ratio.

For becauſe A. B :: C. D, therefore if E ⊏, =, ⊐ G, then is *c* likewiſe F ⊏, =, ⊐ H. c 2. *def.* 5.
therefore it is evident that if G ⊏, =, ⊐ E, then is H ⊏, =, ⊐ F; *d* therefore B. A :: D. C. d 6. *def.* 5.
Which was to be dem.

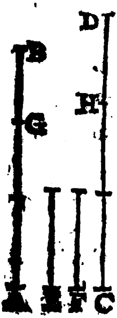
PROP. V.



If a magnitude AB be as multiple of a magnitude CD, as a part taken from the one AE of a part taken from the other CF; the reſidue of the one ſhall be as multiple of the reſidue of the other FD as the whole AB is of the whole CD.

Take any other GA, which ſhall be as multiple to FD the reſidue, as AB is of the whole CD, or as the part taken away AE is of the part
a 1. 5. taken away CF. *a* Therefore the whole GA + AE is as multiple of the whole CF + FD, as the one AE is of the one CF, that is as AB
b 6. *ax.* is of CD. therefore GE *b* = AB; and *c* ſo AE
c 3. *ax.* that was common being taken away, there remains GA = EB.

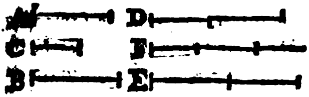
PROP. VI.



If two magnitudes AB, CD be equimultiples of two magnitudes E, F; and ſome magnitudes AG and CH equimultiples of the ſame E, F, be taken away; then the reſidues GB, HD are either equal to thoſe magnitudes E, F, or elſe equimultiples of them.

For becauſe the number of parts in AB, whereof each is equal to E, is put equal to the number of parts in CD, whereof each is equal to F, and alſo the number of parts in AG equal to the number of parts in CH; If from one you take AG, and from the
a 3. *ax.* other CH, *a* then remains the number of parts in the remainder GB equal to the number of parts in HD. therefore if GB be once E, then is HD once F. if GB be many times F, then is HD ſo of F. *Which was to be dem.*

PROP. VII.



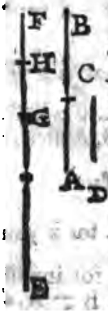
Equal magnitudes A and B have to the ſame magnitude C the ſame proportion or ratio. And one and the ſame magnitude C has the ſame ratio to equal magnitudes A and B.

Take D and E equimultiples of the equal magnitudes A and B, and F any-wiſe multiple of C; then is D *a* = E. Wherefore if D $\sqsubset$, =, a 6. *ax.*
$\sqsupset$ F, then alſo E will be $\sqsubset$, =, $\sqsupset$ F. *b* there- b 6. *def.* 5.
fore A. C :: B. C. and *c* by inverſion C. A :: *c* C. c *cor.* 4. 5.
B. *Which was to be dem.*

Schol.

If inſtead of the multiple F, two equimultiples be taken, it ſhall be the ſame way prov'd that equal magnitudes have the ſame ratio to other magnitudes that are equal between themſelves.

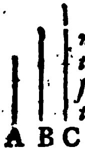
PROP. VIII.



Of unequal magnitudes AB, AC, the greater AB has a greater ratio to the ſame third line D, than the leſſer AC; and the ſame third line D hath a greater ratio to the leſſer AC, than to the greater AB.

Take EF, EG equimultiples of the ſaid AB, AC, ſo that EH being multiple of D be greater than EG, but leſſer than EF. (which will eaſily happen, if both EG and GF be taken greater than D.) It is manifeſt from 8. *def.* 5. that $\frac{AB}{D} \sqsubset \frac{AC}{D}$ and $\frac{D}{AB} \sqsupset \frac{D}{AC}$. *Which was to be dem.*

PROP. IX.

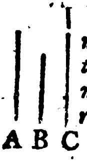


Magnitudes which to one and the ſame magnitude have the ſame ratio, are equal the one to the other. And if a magnitude have the ſame ratio to other magnitudes, thoſe magnitudes are equal one to the other.

1. *Hyp.* If A.C :: B.C; I ſay that $A=B$,
a 8. 5. For let A be greater or leſs than C. *a* then is $\frac{A}{C} \sqsubset$ or $\sqsupset \frac{B}{C}$. *Which is contrary to the Hypotheſis.*

2. *Hyp.* If C. B :: C. A. I ſay that $A=B$. For
b 8. 5. let A be $\sqsubset$ B, *b* then $\frac{C}{B} \sqsubset \frac{C}{A}$. *Which is againſt the Hypotheſis.*

PROP. X.



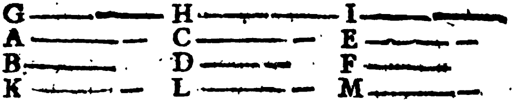
Of magnitudes having ratio to the ſame magnitude, that which has the greater ratio, is the greater magnitude; and that magnitude to which the ſame carries a greater ratio is the leſſer magnitude.

1. *Hyp.* If $\frac{A}{C} \sqsubset \frac{B}{C}$. I ſay that $A \sqsubset B$
a 7. 5. For if it be ſaid that $A=B$, *a* then A.C :: B.C.
b 8. 5. *Which is contrary to the Hyp.* If $A \sqsupset B$, *b* then is $\frac{A}{C} \sqsupset \frac{B}{C}$. *Which is alſo againſt the Hyp.*

2. *Hyp.* If $\frac{C}{B} \sqsubset \frac{C}{A}$ I ſay that $B \sqsupset A$. for if you ſay $B=A$, it's againſt the Hypotheſis, for it will
c 7. 5. *c* follow that C. B :: C. A. If you ſay $B \sqsubset A$, *d*
d 8. 5. then is $\frac{C}{A} \sqsubset \frac{C}{B}$. *Which is alſo againſt the Hyp.*

PROP.

PROP. XI.



Proportions which are one and the ſame to any third, are alſo the ſame one to another.

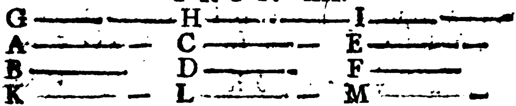
Let A.B :: E. F, and C. D :: E. F. I ſay that A.B :: C.D. Take G, H, I the equimultiples of A, C, E ; and K, L, M the equimultiples of B, D, F. Now *a* becauſe A.B :: E. F, if G ⊏, =, ⊐ K, *b* then after the ſame manner I ⊏, =, ⊐ M. And likewiſe *a* becauſe E. F :: C. D. if I ⊏, =, ⊐ M, *b* then is H likewiſe ⊏, =, ⊐ L. *c* wherefore A. B :: C. D. *Which was to be dem.*

a *hyp.*
b 6. *def.* 3.
c 6. *def.* 5.

Schol.

Proportions that are one and the ſame to the ſame proportions, are the ſame betwixt themſelves.

PROP. XII.



If any number of magnitudes A,B ; C,D ; E and F be proportionals ; as one of the Antecedents A is to one of the Conſequents B ; ſo are all the Antecedents A, C, E to all the Conſequents B, D, F.

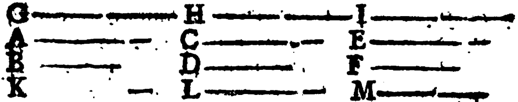
Take the equimultiples of the antecedents G, H, I, and of the conſequents K, L, M. Becauſe that as multiple as one G is of one A, *a* ſo multiples are all G, H, I, of all A, C, E ; and likewiſe as multiple as one K is of one B, ſo multiples are all K, L, M, of all B, D, F. moreover becauſe A. B *b* :: C. D *b* :: E. F. if G be ⊏, =, or ⊐ K. then will H likewiſe be ⊏, =, ⊐ L, and I ⊏, =, ⊐ M. and ſo if G ⊏, =, ⊐ K. in like manner will G + H + I be ⊏, =, ⊐ K + L + M. *c* wherefore A. B :: A + C + E. B + D + F. *Which was to be dem.*

a 1. 5.
b *hyp.*
c 6. *def.* 5.

Coroll.

From hence, if like proportionals be added to like proportionals, the wholes ſhall be proportional.

PROP. XIII.



If the firſt A have the ſame ratio to the ſecond B, that the third C has to the fourth D; and if the third C have a greater proportion to the fourth D, than the fifth E to the ſixth F; then alſo ſhall the firſt A have a greater proportion to the ſecond B, than the fifth E to the ſixth F.

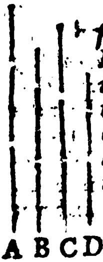
Take G, H, I equimultiples of A, C, E, and K, L, M, equimultiples of B, D, F. Now becauſe
a 6. *def.*5. that A. B :: C. D. if H $\sqsubset$ L. *a* then is G $\sqsubset$ K.
b 8. *def.*5. but becauſe $\frac{C}{D} \sqsubset \frac{E}{F}$, *b* it may be that H $\sqsubset$
c 8. *def.*5. L, and yet I not $\sqsubset$ M. *c* Therefore $\frac{A}{B} \sqsubset \frac{E}{F}$.
Which was to be dem.

Schol.

But if $\frac{C}{D} \sqsupset \frac{E}{F}$, then alſo is $\frac{A}{B} \sqsupset \frac{E}{F}$. Alſo, if $\frac{A}{B} \sqsubset \frac{C}{D} \sqsubset \frac{E}{F}$, then is $\frac{A}{B} \sqsubset \frac{E}{F}$. And if $\frac{A}{B} \sqsupset \frac{C}{D} \sqsupset \frac{E}{F}$, then is $\frac{A}{B} \sqsupset \frac{E}{F}$.

PROP.

PROP. XIV.



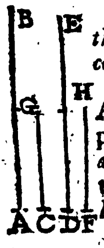
If the ~~first A~~ have the ſame ratio to the ~~ſecond B, that the third C has to the~~ fourth D; and if the firſt A be greater than the third C; then ſhall the ſecond B be greater than the fourth D. But if the firſt A be equal to the third C, then the ſecond B ſhall be equal to the fourth D. but if A be leſſer, the is B alſo leſſer.

Let A $\sqsubset$ C *a* then $\frac{A}{B} \sqsubset \frac{C}{B}$. *b* but a 8. 5.
b *hyp.*
$\frac{A}{B} = \frac{C}{D}$. *c* therefore $\frac{C}{D} \sqsubset \frac{C}{B}$. *c* theref. c 13. 5.
B $\sqsubset$ D. By the like way of argument, if A $\sqsupset$ C,
d then is B $\sqsupset$ D. But if A be put equal to C, d 10. 5.
then C. B :: *e* A. B *f* :: C. D. *g* therefore B = D. e 7. 5.
Which was to be dem. f *hyp.*

Schol. g 11. 5. & 9. 5.

By an argument *à fortiori,* if $\frac{A}{B} \sqsupset \frac{C}{D}$, and A $\sqsubset$ C. then is B $\sqsubset$ D. Likewiſe if A = B, then is B = D. and if A $\sqsubset$, or $\sqsupset$ B, then alſo is C $\sqsubset$ or $\sqsupset$ D.

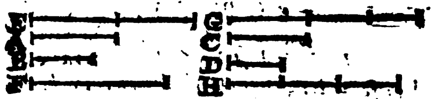
PROP. XV.



Parts C and F are in the ſame ratio, with their like multiples AB and DE, if taken correſpondently. (AB. DE :: C. F.)

Let AG, GB parts of the multiple AB be equal to C; and let DH, HE parts of the multiple DE be equal to F.
a The number of theſe parts is equal to a *hyp.*
the number of thoſe. Therefore whereas b 7. 5.
b AG, C :: DH. F, and GB. C :: HE. F, c 12. 5.
therefore is *c* AG + GB (AB.) DH + HE (DE) :: C. F. *Which was to be dem.*

PROP. XVI.



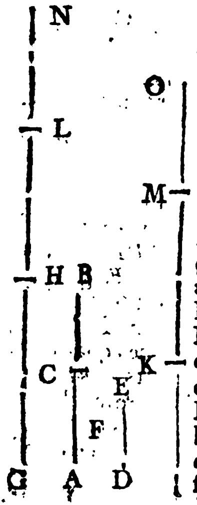
If four magnitudes A, B, C, D be proportional, they alſo ſhall be alternately proportional (A. C :: B. D.)

Take E and F equimultiples of A and B; take
alſo G and H equimultiples of C and D. There-
a 15. 5. fore E. F *a* :: A. B *b* :: C. D *a* :: G. H. Where-
b *hyp.* fore if E ⊏, =, ⊐ G, *c* then likewiſe is F ⊏,
c 11.5.& =, ⊐ H. *d* Therefore A. C :: B. D. *Which was*
14. 5. *to be dem.*
d 6. *def.*5.

Schol.

Alternate ratio has place only then when the quantities are of the ſame kind. For heterogeneous quantities are not compared together.

PROP. XVII.

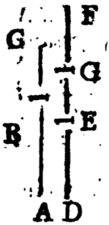


If magnitudes compounded be proportional (AB. CB :: DE. FE.) they ſhall be proportional alſo when divided (AC. CB :: DF. FE.

Take GH, HL, IK, KM, in
order the equimultiples of AC,
CB, DF, FE; and alſo LN,
MO, the equimultiples of CB,
a 1. 5. FE. The whole GL is *a* as
b *conſtr.* multiple of the whole AB, as
one GH of one AC, *b* that is
c 1. 5. as IK of DF, *c* or as the whole
IM of the whole DE. Alſo
d 2. 5. HN (HL + LN) is as *d* multiple
of CB, as KO (KM + MO) is
of FE. Therefore, whereas by
Hyp. AB. BC :: DE. EF. if GL
be ⊏, =, ⊐ HN, then likewiſe
e 6. *def.* *e* will IM ⊏, =, ⊐ KO. Take
from theſe HL, KM that are
equal;

equal; and if the remainder GH be ⊏, =, ⊐ LN, *f* then will IK ⊏, =, ⊐ MO, *g* whence f 5. *ax.*
AC. CB :: DF. EF. *Which was to be dem.* g 6. *def.* 5.

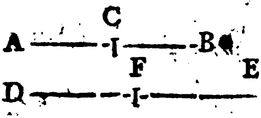
PROP. XVIII.



If magnitudes divided be proportional (AB. BC :: DE. EF) the ſame alſo being compounded ſhall be proportional (AC. CB :: DF. FE.

For if it can be, let AB. CB :: DF. FG ⊐ FE. *a* Then by diviſion will a 17. 5.
AB. BC :: DG. GF. *b* that is, DG, b *hyp.* &
GF :: DE. EF. and being DG ⊏ 11. 5.
DE, *c* therefore is GF ⊏ EF. *d Which* c 14. 5.
is abſurd. The like abſurdity will follow if it d 9. *ax.*
be ſaid AB. CB :: DE. GF ⊏ FE.

PROP. XIX.



If the whole AB be to the whole DE as the part taken away AC is to the part taken away DF, then ſhall the reſidue CB be to the reſidue FE as the whole AB is to the whole DE.

Becauſe *a* AB. DE :: AC. DF, *b* therefore by a *hyp.*
permutation AB. AC :: DE. DF. *c* and thence b 10. 5.
by diviſion AC. CB :: DF. FE. *b* wherefore c 17. 5.
again by permutation AC. DF :: CB. FE. *d* that d *hyp.* &
is, AB. DE :: CB. FE. *Which was to be dem.* 11. 5.

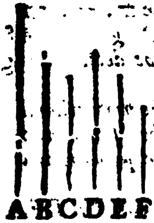
Coroll.

Hence, If like proportionals be ſubſtracted from like proportionals, the reſidues ſhall be proportional.

2. *Hence is converſe ratio demonſtrated.*

Let AB. CB :: DE. FE. I ſay that AB. AC :: DE. DF. For by *a* permutation AB. DE :: CB. a 16. 5.
FE. *b* theref. AB. DE :: AC. DF. whence again b 17. 5.
by permutation AB. AC :: DE. DF. *Which was to be dem.*

PROP.



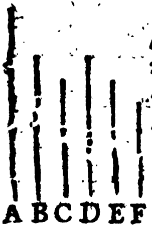
If there be three magnitudes A,B,C, and others D, E, F equal to thoſe in number, which being taken two and two in each order are in the ſame ratio, (A. B :: D. E; and B.C :: E.F,) and if of equality the firſt A be greater than the third C; then ſhall the fourth D be greater than the ſixth F. But if the firſt A be equal to the third C, then the fourth D is ſo to the ſixth F; and if A be leſs than C, ſo D is leſs than F.

a *hyp.* 1. *Hyp.* Let $A \sqsubset C$. Becauſe *a* E.F :: B. C. by *b* in-
b *cor.* 4. 5.
c *hyp. and* 8. 5. verſion ſhall be F.E :: C.B. *c* But $\frac{C}{B} \sqsupset \frac{A}{B}$, therefore
d *ſchol.* 13. 5. $\frac{F}{E} \sqsupset \frac{A}{B}$ or $\frac{D}{E}$, *e* therefore $D \sqsubset F$. *W. W. to be dem.*

e 10. 5. 2. *Hyp.* By the ſame way of argument, if $A \sqsupset C$, it will appear that $D \sqsupset F$.

f 7. 5. 3. *Hyp.* If $A = C$. Becauſe F.E, :: C.B :: *f* A.B ::
g 11. 5. *and* 9. 5. D.E. *g* therefore is D F. *Which was to be dem.*

PROP. XXI.



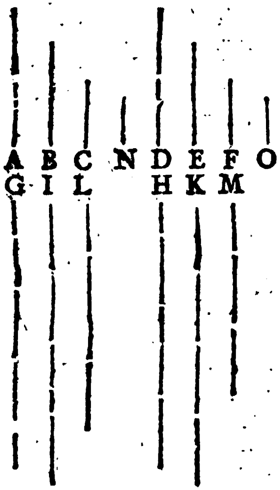
If there be three magnitudes A,B,C, and others alſo D, E, F equal to them in number, which taken two and two are in the ſame ratio; and their proportion inordinate (A. B :: E. F. and B. C :: D.E.) and if of equality, the firſt A be greater than the third C; then is the fourth D greater than the ſixth F: but if the firſt be equal to the third, then is the fourth equal to the ſixth; if leſs, ſo is the other likewiſe.

a *hyp.* 1. *Hyp.* If $A \sqsubset C$; then becauſe *a* D. E :: B. C.
b 8. 5. therefore inverſely E.D :: C.B. but $\frac{C}{B}$ *b* $\sqsupset \frac{A}{B}$ *c* there-
c *ſchol.* 13. 5. fore $\frac{E}{D} \sqsupset \frac{A}{B}$, that is $\frac{E}{F}$, *d* therefore $D \sqsubset F$.
d 10. 5.

2. *Hyp.* By the like arguments, if $A \sqsupset C$, then is $D \sqsupset F$.

3. *Hyp.* If $A = C$; then becauſe E. D :: *e* C. e 7. 5.
B :: *e* A. B :: *f* E. F, *g* therefore is $D = F$. f *hyp.*
Which was to be dem. g 9. 5.

PROP. XXII.

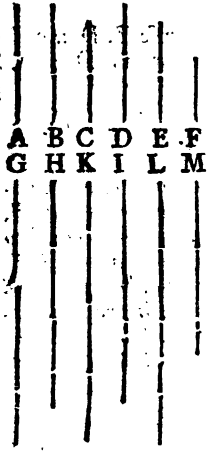


If there be any number of magnitudes A, B, C, and others equal to them in number D, E, F, which taken two and two are in the ſame ratio (A. B :: D. E. and B. C :: E. F) they ſhall be in the ſame ratio alſo by equality, (A. C :: D. F.

Take G. H equimultiples of A, D; and I, K of B, E; and alſo L, M of C, F.

Becauſe *a* A. B :: a *hyp.*
D. E, *b* therefore G. b 4. 5.
I :: H. K. and in like manner I. L :: K. M, therefore, if $G \sqsubset, =, \sqsupset$ L, *c* then is $H \sqsubset, =, \sqsupset$ M. *d* therefore A. c 20. 5.
C :: D. F. By the ſame way of demonſtration if d 6. *def.* 5.
further C. N :: F. O, then by equality A. N :: D. O. *Which was to be dem.*

PROP. XXIII.



If there be three magnitudes A,B,C, and others D, E, F, equal to them in number, which taken two and two are in the ſame ratio, and their proportionality inordinate (A.B :: E.F. and B.C :: D.E.) they ſhall be in the ſame ratio alſo by equality.

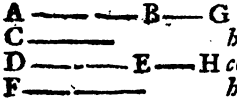
Take G, H, I equimultiples of A,B,D; and alſo K,L,M equimultiples of C,E,F.
Then G.H :: *a* A.B :: *b* E.F
a 15. 5. *a* :: L.M. Moreover becauſe
b *hyp.* *b* B. C :: D.E, thence is *c* H.
c 4. 5. K :: I.L; therefore G,H,K, and I,L,M are according to 21.5. Wherefore if G be ⊏, =, ⊐ K, then is likewiſe I ⊏, =, ⊐ M. and ſo *d* conſequently A.C :: D.F. *Which was to be*
d 6. *def.* 5. *demonſtrated.*

If there be more magnitudes than three, this way of demonſtration holds good in them alſo.

Coroll.

*22.&23. From hence * it follows that ratio's com-
5. *and* 20. pounded of the ſame ratio's are among them-
def. 1. ſelves the ſame; as alſo the ſame parts of the ſame ratio's, are among themſelves the ſame.

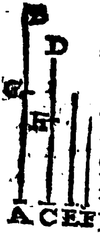
PROP. XXIV.



If the firſt magnitude AB have the ſame ratio to the ſecond C, which the third DE has to the fourth F; and if the fifth BG have the ſame ratio to the ſecond C, which the ſixth EH has to the fourth F, then ſhall the firſt compounded with the fifth (AG) have the ſame ratio to the ſecond C, which the third compounded with the ſixth (DH) has to the fourth F.

For becauſe *a* AB.C :: DE. F, and by the Hyp. a *hyp.*
and inverſion C. BG :: F. EH; therefore by *b* e- b 22. 5.
quality AB. BG :: DE. EH. whence by com-
pounding AG. BG :: DH. EH. Alſo *c* BG. C :: c *hyp.*
EH. F. Therefore again by *b* equality AG. C ::
DH. F. *Which was to be dem.*

PROP. XXV.

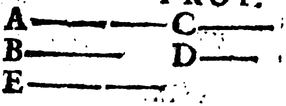


If four magnitudes be proportional (AB. CD :: E. F) the greateſt AB and the leaſt F ſhall be greater than the reſt CD, and E.

Make AG = E, and CH = F. Becauſe
AB. CD :: *a* E. F :: *b* AG.CH. *c* thence a *hyp.*
is AB. CD :: GB.HD. *d* but AB $\sqsubset$ CD. b 7. 5.
e therefore GB $\sqsubset$ HD. But AG + F = c 19. 5.
E + CH, therefore AG + F + GB $\sqsubset$ d *hyp.*
E + CH + HD, that is, AB + F $\sqsubset$ E e *ſchol.* 14.
+ CD. *Which was to be dem.* 5.

Theſe Propoſitions which follow are not Euclide's, *but taken out of other Authors, and here ſubjoined becauſe of their frequent uſe.*

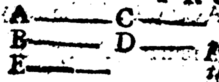
PROP. XXVI.



If the firſt have a greater proportion to the ſecond, than the third to the fourth, then contrarywiſe, by converſion, the ſecond ſhall have a leſs proportion to the firſt, than the fourth to the third.

Let $\frac{A}{B} \sqsubset \frac{C}{D}$. I ſay that $\frac{B}{A} \sqsupset \frac{D}{C}$. For conceive
$\frac{C}{D} = \frac{E}{B}$, *a* therefore $\frac{A}{B} \sqsubset \frac{E}{B}$ *b* whence A $\sqsubset$ E. *c* there- a 13. 5.
fore $\frac{B}{A} \sqsupset \frac{B}{E}$, *d* or $\frac{D}{C}$ *Which was to be dem.* b 10. 5.
c 8. 5.
d *cor.* 4. 5.

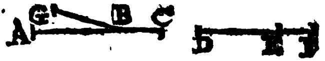
PROP. XXVII.



If the firſt have a greater proportion to the ſecond, than the third to the fourth; then alternately the firſt ſhall have a greater proportion to the third, than the ſecond to the fourth.

Let $\frac{A}{B} \sqsubset \frac{C}{D}$. then I ſay $\frac{A}{C} \sqsubset \frac{B}{D}$. For conceive
a 10. 5. $\frac{E}{B} = \frac{C}{D}$. *a* therefore A$\sqsubset$E, *b* and therefore $\frac{A}{C} \sqsubset \frac{E}{C}$
b 1. 5.
c 16. 5. *c* or $\frac{B}{D}$. *Which was to be dem.*

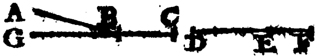
PROP. XXVIII.



If the firſt have a greater proportion to the ſecond than the third to the fourth, then the firſt compounded with the ſecond ſhall have a greater proportion to the ſecond, than the third compounded with the fourth to the fourth.

Let $\frac{AB}{BC} \sqsubset \frac{DE}{EF}$. I ſay that $\frac{AC}{BC} \sqsubset \frac{DE}{EF}$. For con-
a 10. 5. ceive $\frac{GB}{BC} = \frac{DE}{EF}$. *a* therefore is AB$\sqsubset$GB. add BC
b 4. *ax.* to each part. then *b* will AC $\sqsubset$ GC. *c* therefore
c 8. 5.
d 18. 5. $\frac{AC}{BC} \sqsubset \frac{GC}{BC}$. *d* that is $\frac{DF}{EF}$. *Which was to be dem.*

PROP. XXIX.

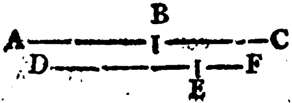


If the firſt compounded with the ſecond have a greater proportion to the ſecond, than the third compounded with the fourth has to the fourth; then by diviſion the firſt ſhall have a greater proportion to the ſecond, than the third to the fourth.

Let $\frac{AC}{BC} \sqsubset \frac{DF}{EF}$ then I ſay $\frac{AB}{BC} \sqsubset \frac{DE}{EF}$. For con-
a 10. 5. ceive $\frac{GC}{BC} = \frac{DE}{EF}$. *a* theref. AC$\sqsubset$GC. Take away
b 5. *ax.* BC, that is common; then *b* remains AB$\sqsubset$GB. *c*
c 8. 4.
d 17. 5. Therefore $\frac{AB}{BC} \sqsubset \frac{GB}{BC}$ *d* or $\frac{DE}{EF}$. *Which was to be dem.*

PROP.

PROP. XXX.



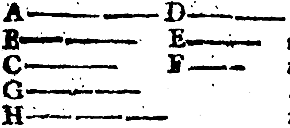
If the firſt compounded with the ſecond have a greater proportion to the ſecond, than the third compounded with the fourth has to the fourth, then by converſe ratio ſhall the firſt compounded with the ſecond have a leſſer ratio to the firſt, than the third compounded with the fourth ſhall have to the third.

Let $\frac{AC}{BC} \sqsubset \frac{DF}{EF}$. Then I ſay that $\frac{AC}{AB} \sqsupset \frac{DF}{DE}$.

For becauſe that $\frac{AC}{BC}$ *a* $\sqsubset \frac{DF}{EF}$, *b* therefore by diviſion $\frac{AB}{BC} \sqsubset \frac{DE}{EF}$ by converſion *c* therefore $\frac{BC}{AB} \sqsupset \frac{EF}{DE}$ and *d* therefore by compoſition $\frac{AC}{AB} \sqsupset \frac{DF}{DE}$. *Which was to be dem.*

a *hyp.*
b 29. 5.
c 26. 5.
d 28. 5.

PROP. XXXI.



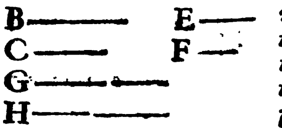
If there be three magnitudes A, B, C, and others alſo D, E, F equal to them in number; and if there be a greater proportion of the firſt of the former to the ſecond, than there is of the firſt of the laſt to their ſecond ($\frac{A}{B} \sqsubset \frac{D}{F}$;) *and there be alſo a greater proportion of the ſecond of the firſt magnitudes to the third, than there is of the ſecond of the laſt magnitude to their third* ($\frac{B}{C} \sqsubset \frac{E}{F}$.) *Then by equality alſo ſhall the ratio of the firſt of the former magnitudes to the third be greater than the ratio of the firſt of the latter magnitudes to the third* ($\frac{A}{C} \sqsubset \frac{D}{F}$.)

Con-

a 10. 5.
b 8. 5.
c 13. 5.
d 10. 5.
e 8. 5.
f 12. 5.

Conceive $\frac{G}{C} = \frac{E}{F}$. *a* therefore is $E \sqsubset G$, and *b* therefore $\frac{A}{G} \sqsubset \frac{A}{B}$. Again conceive $\frac{H}{G} = \frac{D}{F}$. *c* therefore $\frac{H}{G} \sqsupset \frac{A}{B}$, therefore much more $\frac{H}{G} \sqsupset \frac{A}{G}$. *d* wherefore $A \sqsubset H$, *e* and conſequently $\frac{A}{C} \sqsubset \frac{H}{C}$ *f* or $\frac{D}{F}$.

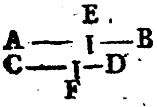
P R O P. XXXII.



If there be three magnitudes A, B, C, and others D, E, F, equal to them in number; and there be a greater proportion of the firſt of the former magnitudes to the ſecond, than there is of the ſecond of the latter to the third $\left(\frac{A}{B} \sqsubset \frac{E}{F}\right)$ *and alſo the ratio of the ſecond of the former to the third be greater than the ratio of the firſt of the latter to the ſecond* $\left(\frac{B}{C} \sqsubset \frac{D}{E}.\right)$ *then by equality alſo ſhall the proportion of the firſt of the former to the third, be greater than that of the firſt of the latter to the third* $\left(\frac{A}{C} \sqsubset \frac{D}{E}.\right)$

The demonſtration of this Propoſition is altogether like that of the precedent.

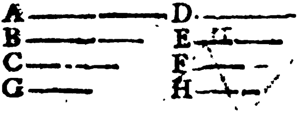
P R O P. XXXIII.



If the proportion of the whole AB to the whole CD be greater than the proportion of the part taken away AE to the part taken away CF; then ſhall alſo the ratio of the remainder EB to the remainder FD be greater than that of the whole AB to the whole CD.

Becauſe that $\frac{AB}{CD}$ *a* $\sqsubset \frac{AE}{EF}$. *b* therefore by permutation $\frac{AB}{AE} \sqsubset \frac{CD}{CF}$, *c* therefore by converſe ratio $\frac{AB}{EB} \sqsupset \frac{CD}{FD}$, and by permutation again $\frac{AB}{CD} \sqsupset \frac{EB}{FD}$.

PROP. XXXIV.



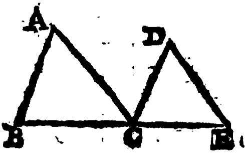
If there be any number of magnitudes, and others alſo equal to them in number; and the proportion of the firſt of the former to the firſt of the latter, be greater than that of the ſecond to the ſecond, and that greater than the proportion of the third to the third, and ſo forward: all the former magnitudes together ſhall have a greater ratio to all the latter together, than all the former, leaving out the firſt, ſhall have to all the latter, leaving out the firſt; but leſs than that of the firſt of the former to the firſt of the latter; and laſtly greater than that of the laſt of the former to the laſt of the latter.

You may pleaſe to conſult Interpreters for the demonſtration hereof, we having for brevity ſake omitted it, and becauſe 'tis of no uſe in theſe Elements.

The End of the fifth Book.

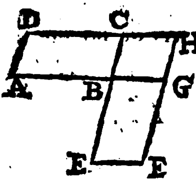
THE SIXTH BOOK OF EUCLIDE's ELEMENTS.

Definitions.

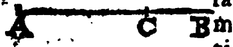


I. LIke right-lined figures (ABC, DCE) are ſuch whoſe ſeveral angles are equal one to the other, and alſo their ſides about the equal angles, proportional.

The angle B = DCE, and AB. BC :: DC. CE. Alſo the angle A = D, and BA. AC :: CD. DE. Laſtly the angle ACB = E, and BC. CA :: CE. ED.

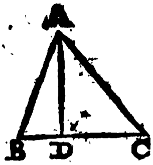


II. Reciprocal figures are (BD, BF) when in either figure are the terms antecedents and conſequents of ratio's, that is, AB. BG :: FB. BC.)



III. A right line AB is ſaid to be cut according to mean and extreme proportion, when as the whole AB is to the greater ſegment AC, ſo is the greater ſegment AC to the leſs CB (AB. AC :: AC. CB.)

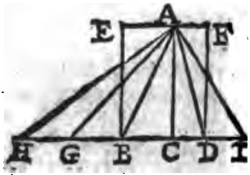
IV. The



IV. The altitude of any figure ABC, is a perpendicular line AD drawn from the top A to the baſe BC.

V. A ratio is ſaid to be compounded of two ratio's, when the quantities of the ratio's being multiplied the one into the other, do produce any ratio. *As the ratio of A to C is compounded of the ratio's of A to B and B to C. For* $\frac{A}{B} + \frac{B}{C} \,{}^{a}\!= \frac{A}{C} \,{}^{b}\!= \frac{AB}{BC}$. a 10. *def.* 5. b 15. 4.

PROP. I.

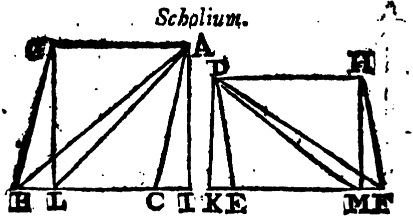


Triangles ABC, ACD, and parallelograms BCAE, CDFA which have the ſame height, are in proportion one to the other, as their baſes BC, CD are.

a Take as many as you pleaſe, BG, GH, equal to BC, and alſo DI = CD. and join AG, AH, AI. a 3. 1.

b The triangles ACB, ABG, AGH are equal, and *b* alſo the triangle ACD = ADI. Therefore the triangle ACH is as multiple of the triangle ACB, as the baſe HC is of the baſe BC; and the triangle ACI as multiple of the triangle ACD, as the baſe CI is of CD. But if HC $\sqsubset$, $=$, $\sqsupset$ CI, *c* then is likewiſe the triangle AHC $\sqsubset$, $=$, $\sqsupset$ ACI; and *d* therefore BC. CD :: the triangle ABC. ACD :: *e* Pgr. CE. GF. *Which was to be dem.* b 38. 1. c *ſch.* 38. 1. d 6. *def.* 5. e 41. 1. & 15. 5.

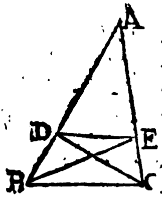
Scholium.



Hence, *Triangles ABC, DEF, and Pgrs. AGBC, DEFH, whoſe baſes BC, EF are equal, are in ſuch proportion as their altitudes, AI, DK are.*

a 3. 1. *a* Take IL = CB, and KM = EF; and join
b 7. 5. LA, LG, MD, MH. then is it evident that the
c 1. 6. triangles ABC.DEF :: *b* ALI.DKM :: *c* AI.DK ::
d 41. 1. & 15. 5. *d* pgrs. AGBC. DEFH. *Which was to be dem.*

PROP. II.



If to one ſide BC of a triangle ABC be drawn a parallel right line DE, the ſame ſhall cut the ſides of the triangle proportionally (AD. BD :: AE. EC.) And if the ſides of the triangle be proportionally cut (AD. BD :: AE. EC) then a right line DE joined at the ſections D, E ſhall be parallel to the remaining ſide of the triangle BC.

Draw CD and BE.

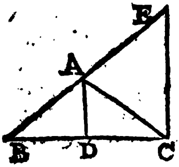
a 17. 1. 1. *Hyp.* Becauſe the triangle DEB *a* = DEC,
b 7. 5. *b* therefore ſhall be the triangle ADE. DBE ::
c 2. 6. ADE. ECD. But the triangle ADE. DBE :: *c*
d 11. 5. AD. DB, and the triangle ADE. DEC :: AE.
EC, *d* therefore AD. DB :: AE. EC.

e 1. 6. 2. *Hyp.* Becauſe AD. DB :: AE. EC, *e* that is
f 19. 5. the triangle ADE. DBE :: ADE. ECD; *f* there-
g 39. 1. fore is the triangle DBE = ECD; and *g* therefore DE, BC are parallels. *Which was to be dem.*

Scholium.

If there be drawn many parallels to one ſide of any triangle, then all the ſegments of the ſides ſhall be proportional; as is eaſily deducible from the precedent.

PROP. III.



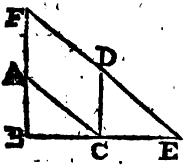
If an angle BAC of a triangle BAC be biſected, and the right line AD, that biſects the angle, cut the baſe alſo; then ſhall the ſegments of the baſe have the ſame ratio that the other ſides of the triangle have (BD. DC :: AB. AC.) And if the ſegments of the baſe have the ſame ratio, that the other ſides of the triangle have (BD. DC :: AB. AC.) then a right line AD drawn from the top A to the ſection D, ſhall biſect that angle BAC of the triangle.

Produce BA, and make AE = AC, and join CE.

1. *Hyp.* Becauſe AE = AC, therefore is the
angle ACE *a* = E *b* = $\frac{1}{2}$ BAC *c* = DAC; *d* a 5. 1.
therefore DA, CE are parallels. *e* Wherefore b 32. 1.
BA. AE (AC) :: BD. DC. c *hyp.*

2. *Hyp.* Becauſe BA. AC (AE) :: BD. DC, *f* d 27. 1.
therefore are DA, CE parallels; and *g* therefore e 2. 6.
is the angle BAD = E; and the angle DAC *g* f 2. 6.
= ACE *h* = E. *k* therefore the angle BAD = g 29. 1.
DAC. Wherefore the angle BAC is biſected. h 5. 1.
Which was to be dem. k 1. *ax.*

PROP. IV.



Of equiangular triangles ABC, DCE, the ſides are proportional which are about the equal angles, B, DCE, (AB. BC :: DC. CE, &c.) And the ſides AB, DC, &c. which are ſubtended under the equal angles ACB, E, &c. are homologous, or of like ratio.

Set the ſide BC in a direct line to the ſide CE, and produce BA and ED till they *a* meet. a 32. 1.

b *hyp.* Becauſe the angle B *b* = ECD, *c* therefore
c 28. 1. BF, CD are parallel; Alſo becauſe the angle
BCA *b* = CED, *c* therefore are CA, EF parallel.
d 34. 1. Therefore the figure CAFD is a Pgr. *d* therefore
AF = CD, and AC = *d* ED. Whence it is evi-
e 2. 6. dent that AB. AF (CD) :: *e* BC. CE. *f* by permu-
f 16. 5. tation therefore AB. BC :: CD. CE. alſo BC.
CE :: FD (AC.) DE. *f* and thence by per-
g 22. 5. mutation BC. AC :: CE. DE. *g* Wherefore alſo
by equality AB. AC :: CD. DE. Therefore, *&c.*

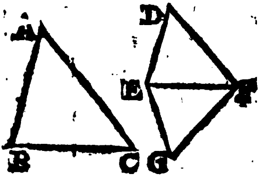
Coroll.

Hence AB. DC :: BC. CE :: AC. DE.

Schol.

Hence, If in a triangle FBE there be drawn AC a parallel to one ſide FE, the triangle ABC ſhall be like to the whole FBE.

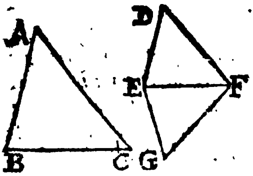
PROP. V.



If two triangles ABC, DEF have their ſides proportional (AB. BC :: DE. EF, and AC. BC :: DF. EF, and alſo AB. AC :: DE. DF) thoſe triangles are equiangular, and thoſe angles equal under which are ſubtended the homologous ſides.

a 23. 1. At the ſide EF *a* make the angle FEG = B,
b 32. 1. and the angle EFG = C; *b* whence the angle
c 4. 6. G = A. Therefore GE. EF *c* :: AB. BC :: DE.
d *hyp.* EF. *e* and therefore GE = DE. Likewiſe GF.
e 11. 5. *and* EF *c* :: AC. CB :: *d* DF. FE. *e* therefore GF =
9. 5. DF. Therefore the triangles DEF, GEF are
f 8. 1. mutually equilateral. *f* Therefore the angle D =
g 32. 1. G = A, and the angle FED *f* = FEG = B, and
g conſequently the angle DEF = C. Therefore, *&c.*

PROP. VI.

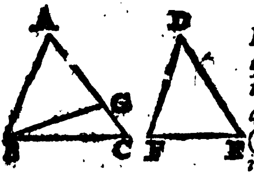


If two triangles ABC, DEF have one angle B equal to one angle DEF, and the ſides about the equal angles B, DEF proportional (AB.BC :: DE. EF) then thoſe triangles ABC, DEF are equiangular, and have thoſe angles equal, under which are ſubtended the homologous ſides.

At the ſide EF make the angle FEG = B, and
the angle EFG = C; *a* then will the angle G = A. a 32. 1.
Therefore GE.EF :: *b* AB. BC :: *c* DE.EF, *d* and b 4. 6.
therefore DE = GE. But the angle DEF *e* = B *f* = c *hyp.*
GEF; therefore the angle D *g* = G = A, *h* and d 9. 5.
conſequently the angle EFD = C. *W.W. to be dem.* e *hyp.*

PROP. VII.

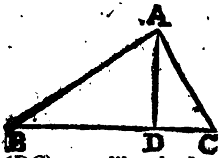
f *conſtr.*



If two triangles ABC, g 4. 1.
DEF have one angle A e- h 32. 1.
qual to one angle D, and the ſides about the other angles ABC, E, proportional (AB. BC :: DE. EF) and if they have both of the remaining angles C, F either leſs or not leſs than a right angle; then ſhall the triangles ABC, DEF be equiangular, and have thoſe angles equal about which the proportional ſides are.

For, if it can be, let the angle ABC $\sqsubset$ E, and
make the angle ABG = E. Therefore, whereas
the angle A *a* = D, *b* thence is the angle AGB a *hyp.*
= F. Therefore AB. BG *c* :: DE. EF :: *d* AB. b 32. 1.
BC. *e* therefore BG = BC. *f* therefore the angle c 4. 6.
BGC = BCG. *g* Therefore BGC or C is leſs d *hyp.*
than a right angle, and *h* conſequently AGB or e 9. 5.
F is greater than a right: Therefore the angles f 5. 1.
C and F are not of the ſame ſpecies or kind, g *cor.*17.1.
which is againſt the Hypotheſis. h *cor.*12.1.

PROP. VIII.



If in a right-lined triangle ABC, from the right angle BAC there be drawn AD a perpendicular to the baſe BC; then the triangles about the perpendicular (ADB, ADC) are like both to the whole triangle ABC, and alſo one to the other.

a hyp. b 12. ax. c 32. & 4. 6 d 21. 6. For becauſe BAC, ADB are *a* right angles, *b* and ſo equal, and B common; the triangles BAC, ADB *c* are like. By the ſame argument BAC, ADC are like *d* whence alſo ADB, ADC will be like. *Which was to be dem.*

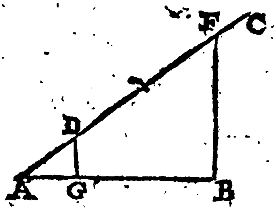
e 1. def. 8.

Coroll.

Hence, 1. BD. DA *e*:: DA. DC.

2. BC. AC :: AC. DC. and CB. BA :: BA. BD.

PROP. IX.



From a right line given AB to cut off any part required, as $\frac{1}{3}$ (AG.)

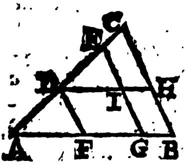
a 31. 1. From the point A draw an infinite line AC any-wiſe, in which, *a* take any three equal parts AD, DE, EF. join FB. to which from D *b* draw the parallel DG. *And the thing is done.*

b 31. 1.

c 2. 6. d 28. 5. For GB. AG :: *c* FD. AD; whence by *d* compoſition AB. AG :: AF. AD. therefore, whereas AD = $\frac{1}{3}$ of AF, therefore is AG = $\frac{1}{3}$ of AB. *Which was to be done.*

PROP.

PROP. X.

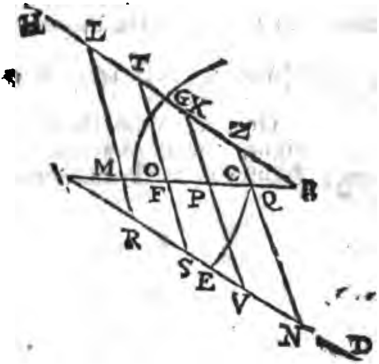


To divide a right line given AB not divided (in F and G) *as another line given AC was cut* (in D and E.)

Let a right line BC join the extremities of the line divided, and of the line not divided; and to that line from the points E, D *a* draw the parallels EG, DF meeting with the right line that is to be cut, in G and F; then the thing is done. a 31. 1.

For let DH be *a* drawn parallel to AB. Then AB. DE :: *b* AF. FG. and DE. EC *b* :: DI. IH :: *c* FG. GB. *Which was to be done,* b 2. 6. c 34. 1. & 7. 5.

Schol.



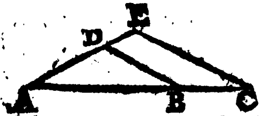
Hence is learnt to cut a right line given AB into as many equal parts as you pleaſe (ſuppoſe 5;) which will be more eaſily performed thus.

Draw an infinite line AD, and another BH parallel to it, and infinite alſo. Of thoſe take equal parts, AR, RS, SV, VN; and BZ, ZX, XT, TL;

in each line leſs parts by one, than are required in AB; then let the right lines LR, TS, XV, ZN be drawn; theſe lines ſo drawn ſhall cut the right line given AB into five parts.

a 33. 1. For RL, ST, VX, NZ are *a* parallels; there-
b *conſtr.* fore, whereas AR, RS, SV, VN are *b* equal; *c*
c 2. 6. thence AM, MO, OP, PQ are equal alſo. Likewiſe, becauſe that BZ = ZX, therefore is BQ = PQ. and therefore AB is cut into five parts. *Which was to be dem.*

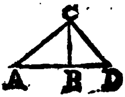
PROP. XI.



Two right lines being given AB, AD, to find out a third in proportion to them (DE.)

Join BD, and from AB being produced take BC = AD. Thro' C draw CE parallel to DB; with which let AD produced meet in E. then is DE the proportional required.

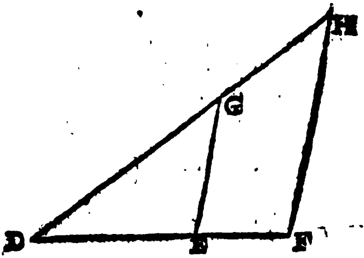
a 2. 6. For AB. BC (AD) *a* :: AD. DE. *Which was to be dem.*



Or thus: Make the angle ABC right, and alſo the angle ACD
b *cor.* 9. right, then *b* AB. BC :: BC. BD.

PROP.

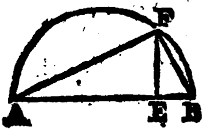
PROP. XII.



Three right lines being given DE, EF, DG to find out a fourth proportional GH.

Join EG, and thro' F draw FH parallel to EG; with which let DG produced to H meet. Then it is evident that DE. EF *a* :: DG. GH. a 2. 6. *Which was to be done.*

PROP. XIII.



Two right lines being given AE, EB, to find out a mean proportional EF.

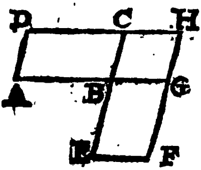
Upon the whole line AB as a diameter deſcribe a ſemicircle AFB, and from E erect a perpendicular EF meeting with the periphery in F. then AE. EF :: EF. FB. For let AF and FB be drawn; *a* then from the right angle of the a 31. 1. right-angled triangle AFB is drawn a right line FE perpendicular to the baſe. *b* Therefore AE. b *cor.* 8. 6. FE :: FE. EB. *Which was to be done.*

Coroll.

Coroll.

Hence, A right line drawn in a circle from any point of the diameter perpendicularly, and extended to the circumference, is a mean proportional betwixt the two ſegments of the diameter.

PROP. XIV.

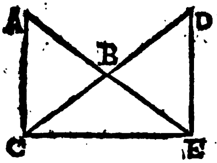


Equal Parallelograms having one angle ABC equal to one EBG, have the ſides BD, BF which are about the equal angles reciprocal (AB. BG :: EB. BC;) And thoſe Parallelograms BD, BF which have one angle ABC equal to one EBG, and the ſides which are about the equal angles reciprocal, are equal.

For let the ſides AB, BG about the equal an-
a *ſch.*15.1. gles make one right line; *a* wherefore EB, BC
b 1. 6. ſhall do the ſame. Let FG, DC be produced
c 7. 5. till they meet.
d 1. 6. 1. *Hyp.* AB. BG *b* :: *f* BD. BH :: *c* BF. BH ::
e 11. 5. *d* BE. BC. *e* therefore, &c.
f 1. 6. 2. *Hyp.* BD. BH :: *f* AB. BG :: *g* BE. BC :: *h*
g *hyp.* BF. BH. *k* Therefore the Pgr. BD=BF. *Which*
h 1. 6. *was to be dem.*
k 11. *and*
9. 5.

PROP.

PROP. XV.

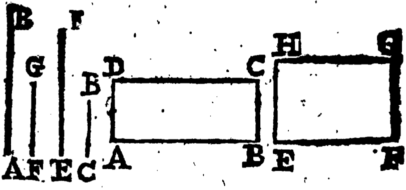


Equal triangles having one angle ABC equal to one DBE, their ſides which are about the equal angles are reciprocal (AB. BE :: DB. BC.) And thoſe triangles that have one angle ABC equal to one DBE, and have alſo the ſides that are about the equal angles reciprocal (AB. BE :: DB. BC) are equal.

Let the ſides CB, BC, which are about the
equal angles be ſet in a ſtrait line; *a* therefore a *ſch.*15.1.
ABE is a right line. Let CE be drawn.
1. *Hyp.* AB.BE :: *b* the triangle ABC.CBE *c* :: b 1. 6.
the triangle DBE.CBE :: *d* DB.BC. *e* therefore, &c. c 7. 5.
1. *Hyp.* The triangle ABC. CBE :: *f* AB. d 1. 6.
BE :: *g* DB. BC *h* :: the triangle DBC. CBE. *k* e 11. 5.
Therefore the triangle ABC = DBE. *Which* f 1. 6.
was to be dem. g *hyp.*

h 1. 6.
k 11, *and* 9. 5.

PROP. XVI.



If four right lines be proportional (AB. FG :: EF. CB) the rectangle AC comprehended under the extremes AD, CB, is equal to the rectangle EG comprehended under the means FG, EF. And if the rectangle AC comprehended under the extremes AB, CB be equal to the rectangle EG comprehended under the means FG, EF, then are the four right lines proportional, (AB. FG :: EF. CB.)

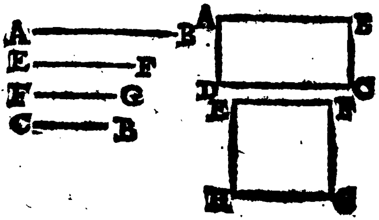
1. *Hyp.*

a 12. *ax.* 1. *Hyp.* The angles B and F are right, and *a* conſequently equal, and by Hyp. AB. FG :: EF. CB.
b 14. 6. *b* therefore the rectangle AC = EG.
c *hyp.* 2. *Hyp.* The rectangle AC *c* = EG, and the
d 14. 6. angle B = F; *d* therefore AB. FG :: EF. CB.
Which was to be dem.

Coroll.

Hence, it is eaſy to apply a rectangle given EG to a right line given AB; (*viz.*) *e* by making AB. EF :: FG. BC.
e 12. 6.

PROP. XVII.



If three right lines be proportional (AB. EF :: EF. CB.) the rectangle AC made under the extremes AB, CB is equal to the ſquare EG made of the middle EF. And if the rectangle AC comprehended under the extremes AB, CB be equal to the ſquare EG made of the middle EF, then the three lines are proportional, (AB. EF :: EE. CB.)

Take FG = EF.

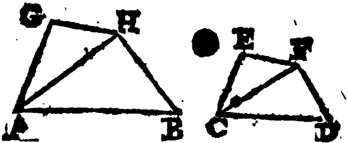
a *hyp.* 1. *Hyp.* AB. EF :: *a* EF (FG.) CB. therefore
b 16. 6. the rectangle AC *b* = EG *c* = EFq.
c 29. *def.* 1. 2. *Hyp.* The rectangle AC *d* = to the ſquare
d *hyp.* EG = EFq. *e* therefore AB. EF :: FG (EF.) BC.
e 16. 6. *Which was to be dem.*

Coroll.

Let A × B = Cq. therefore A. C :: C. B.

PROP.

PROP. XVIII.

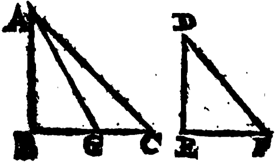


From a right line given AB to deſcribe a right-lined figure AGHB like and alike ſituate to a right-lined figure given CEFD.

Reſolve the right-lined figure given into triangles; *a* Make the angle ABH = D, *a* and a 23. 1.
the angle BAH = DCF, *a* and the angle AHG = CFE, *b* and the angle HAG = FCE. then AGHB ſhall be the right-lined figure ſought.

For the angle B *b* = D, and the angle BAH b *conſtr.*
b = DCF. *c* wherefore the angle AHB = CFD. c 32. 1.
b alſo the angle HAG = FCE; and the angle AHG *b* = CFE. *c* wherefore the angle G = E,
and the whole angle GAB *d* = ECD, and the d 2. *ax.*
whole angle GHB *d* = EFD. The Polygones therefore are mutually equiangular. Moreover becauſe the triangles are equiangular, therefore
AB. BH *e* :: CD. DF; and AG. GH *e* :: CE. EF. e 4. 6.
Likewiſe AG. AH :: *e* CE. CF. and AH. AB :: f 22. 5.
CF. CD. *f* From whence by equality, AG. AB :: g 6. *def.* 5.
CF. CD. After the ſame manner GH. HB :: EF. FD. Therefore the Polygones ABHG, CDEF are like and alike ſituate. *Which was to be done.*

PROP. XIX.



Like triangles ABC, DEF are in duplicate ratio of their homologous ſides, BC, EF.

a Let there be a 11. 6.
made BC, EF :: EF. BG. and let AG be drawn. Becauſe that AB.

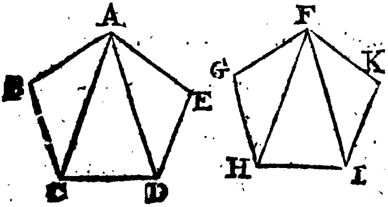
DE

b *cor.*4.6. DE *b* :: BC. EF *c* :: EF. BG, and the angle B=E.
c *conſtr.* *d* therefore is the triangle ABG=DEF. But the
d 15.6.
e 1.6. triangle ABC. ABG :: *e* BC. BG, and *f* $\frac{BC}{BG}=\frac{BC}{EF}$
f 10.*def.*5.
g 11.5. twice; therefore $\frac{ABC}{ABG}$ that is $\frac{ABC}{DEF}$ *g* = $\frac{BC}{EF}$ twice.
Which was to be dem.

Coroll.

Hence, If three right lines (BC, EF, BG) be proportional, then as the firſt is to the third, ſo is a triangle made upon the firſt BC, to a triangle like and alike deſcribed upon the ſecond EF: or ſo is a triangle deſcribed upon the ſecond EF to a triangle like and alike deſcribed upon the third.

PROP. XX.



Like Polygones ABCDE, FGHIK are divided into equal triangles ABC, FGH, and ACD, FHI, and ADE, FIK; both equal in number, and homologous to the wholes (ABC. FGH :: ABCDE. FGHIK :: ACD. FHI :: ADE. FIK.) And the Polygones ABCDE, FGHIK have a double ratio one to the other what one homologous ſide BC has to the other homologous ſide GH.

a *hyp.* 1. For the angle B *a*=G, and AB. BC *a* :: FG.
b 6.6. GH. *b* therefore the triangles ABC, FGH are

are equiangular. After the ſame manner are the
triangles AED, FKI like. Whereas therefore
the angle BCA *b* = GHF, and the angle ADE b 6. 6.
b = FIK, and the whole angles BCD, GHI,
and the whole angles CDE, HIK are *c* equal, c *hyp.*
there remains the angle ACD *d* = FHI, and the d 3. *ax.*
angle ADC = FIH. *e* from whence alſo the e 32. 1.
angle CAD = HFI. therefore the triangles
ACD, FHI are like. Therefore, *&c.*

2. Becauſe that the triangles BCA, GHF are like,
f therefore is $\frac{BCA}{GHF} = \frac{BC}{GH}$ twice. For the ſame rea- f 19. 6.
ſon is $\frac{CAD}{HFI} = \frac{CD}{HI}$ twice; laſtly $\frac{DEA}{IKF} = \frac{DE}{IK}$
twice. now whereas that BC. GH *g* :: CD. HI g *hyp. and*
:: DE. IK. *h* therefore is the triangle BCA. 16. 5.
GHF :: CAD. HFI :: DEA. IKF *k* :: the poly- h *cor.* 23. 5.
gone ABCDE. FGHIK :: $\frac{BC}{GH}$ twice. k 12. 5.

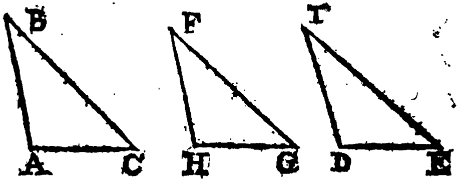
Coroll.

1. Hence, If there be three right lines proportional, then as the firſt is to the third, ſo is a polygone made upon the firſt to a polygone made on the ſecond, like and alike deſcribed; or ſo is a polygone deſcribed upon the ſecond to a polygone made on the third like and alike deſcribed.

By which is found out a method of enlarging or diminiſhing any right-lined figure in a ratio given:
As if you would make a pentagone quintuple of
that pentagone whereof CD is the ſide, then betwixt AB and 5 AB find out a mean proportional * upon this raiſe a pentagone like to * 18. 6.
that given, and it ſhall be quintuple of the pentagone given.

2. Hence alſo, If the homologous ſides of like figures be known, then will the proportion of the figures be evident, *viz.* by finding out a third proportional.

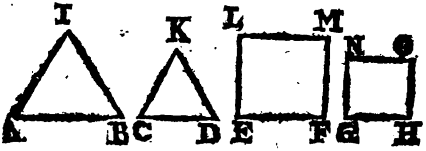
PROP. XXI.



Right lined figures ABC, DIE which are like to the ſame right-lined figure HFG, are alſo like one to the other.

a 1. def.6. For the angle A $\overset{a}{=}$ H $\overset{a}{=}$ D; and the angle C $\overset{a}{=}$ G $\overset{a}{=}$ E; and the angle B $\overset{a}{=}$ F $\overset{a}{=}$ I. Alſo a AB.AC :: HF.HG :: DI.DE. & a AC.CB :: HG.GF :: DE.EI. And AB.BC :: HF.FG :: DI.IE. Therefore ABC, DIE are alike. *Which was to be dem.*

PROP. XXII.



If four right lines be proportional (AB.CD :: EF. GH) the right-lined figures alſo deſcribed upon them being like and in like ſort ſituate, ſhall be proportional (ABI. CDK :: EM. GO.) And if the right-lined figures deſcribed upon the lines, like and alike ſituate, be proportional (ABI. CDK :: EM. GO.) then the right lines alſo ſhall be proportional (AB.CD :: EF, GH.)

a 19. 6. 1. *Hyp.* $\frac{ABI}{CDK} \overset{a}{=} \frac{AB}{CD}$ twice $\overset{a}{=} \frac{EF}{GH}$ twice $\overset{a}{=} \frac{EM}{GO}$.

b hyp. *b* therefore ABI. CDK :: EM. GO.

c 20. 6.

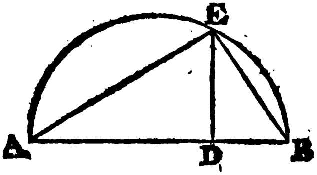
d cor.23.5. 2. *Hyp.* $\frac{AB}{CD}$ twice $\overset{a}{=} \frac{ABI}{CDK} \overset{b}{=} \frac{EM}{GO} \overset{c}{=} \frac{EF}{GH}$ twice. Therefore AB, CD :: EF, GH. *Which was to be dem.*

Schol.

Schol.

Hence is deduced the manner and reaſon of multiplying ſurd quantities. ex. gr. Let $\surd 5$ to be multiplied into $\surd 3$. I ſay that the product will be $\surd 15$. For by the definition of multiplication it ought to be, as 1. $\surd 3$ :: $\surd 5$. to the product. Therefore by this q. 1. q. $\surd 3$ :: q. $\surd 5$. q. of the product. therefore the ſquare of the product is 15. Wherefore $\surd 15$ is the product of $\surd 3$ into $\surd 5$. *Which was to be dem.*

THEOREM.



If a right line AB be cut any-wiſe in D, the rectangle comprehended under the parts AD, DB is a mean proportional betwixt their ſquares. Likewiſe the rectangle comprehended under the whole AB and one part AD, or DB, is a mean proportional betwixt the ſquare of the whole AB and the ſquare of the ſaid part AD, or DB. *Pet.Herig.*

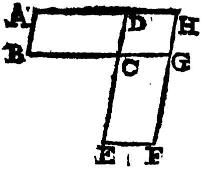
Upon the diameter AB deſcribe a ſemicircle; from D erect a perpendicular DE meeting with the periphery in E. join AE. BE.

It's evident that AD. DE *a* :: DE. DB. *b* there- a *cor.* 8. 6.
fore ADq DEq :: DEq. DBq. *c* that is, ADq. b 22. 6.
ADB :: ADB. DBq. *Which was to be dem.* c 17. 6.

Moreover BA. AE :: *d* AE. AD. *e* therefore BAq. d *cor.* 8. 6.
AEq :: AEq. ADq. *f* that is, BAq. BAD :: BAD. e 22. 6.
ADq. After the ſame manner ABq. ABD :: ABD. f 17. 6.
BDq. *Which was to be dem.*

Or thus: ſuppoſe Z = A + E. It is manifeſt that
Aq. AE :: *a* A. E :: *a* AE. Eq. alſo Zq. ZA :: *a* a 1. 6.
Z. A :: ZA. Aq. and Zq. ZE :: *a* Z. E :: ZE. Eq.

PROP. XXIII.



Equiangular parallelograms AC, CF, have the ratio one to the other, which is compounded of their ſides. $\left(\frac{AC}{CF} = \frac{BC}{CG} + \frac{DC}{CF}\right)$

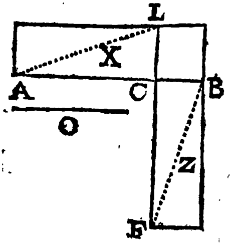
Let the ſides about the equal angles C be *a* ſet in a direct line, and let the Pgr. CH be completed. Then is the ratio of $\frac{AC}{CF}\ {}^{b}\!= \frac{AC}{CH} + \frac{CH}{CF}\ {}^{c}\!= \frac{BC}{CG} + \frac{DC}{CE}$ *Which was to be dem.*

a ſch. 15.
b 20. def. 5.
c 1. 6.

Coroll.

Andr. Tacq. 15. 5.

Hence, *and from* 34. 1. *it appears,* 1. *That triangles which have one angle equal (as* C*) have a ratio compounded of the ratio's of the right lines(AC to CB, and LC to CF) containing the equal angle.*



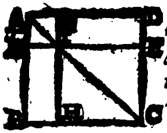
* 35. 1.

2. *That all rectangles, and * conſequently all parallelograms, have their ratio one to the other compounded of the ratio's of baſe to baſe, and altitude to altitude.* After the like manner you may argue in triangles.

3. *From hence is apparent how to give the proportion of triangles and parallelograms.* Let there be two Pgrs. X and Z, whoſe baſes are AC, CB, and altitudes CL, CF. Make CL. CF :: CB. O. * then will it be X.Z :: AC. O.

* 14. 6. *and* 1, 6.

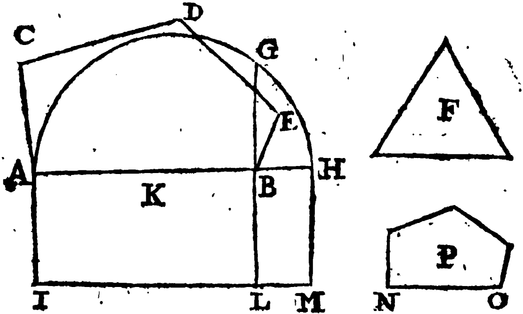
PROP. XXIV.



In every parallelogram ABCD, the parallelograms EG, HF which are about the diameter AC are like to the whole, and alſo one to the other.

For the Pgrs. EG, HF have
each of them one angle common with the whole,
a therefore they are equiangular to the whole, a 29. 1.
and alſo one to the other. Alſo both the triangles ABC, AEI, IHC *a* and the triangles ADC,
AGI, IFC are equiangular mutually; *b* therefore b 4. 6.
AE. EI :: AB, BC, and *b* AE. AI :: AB, AC, and
b AI. AG :: AC. AD. *c* Therefore by equality, c 22. 5.
AE. AG :: AB. AD. *d* Therefore the Pgrs. EG, d 1. *def.* &
BD are like. After the ſame manner are HF, 6.
BD like alſo. Therefore, *&c.*

PROP. XXV.

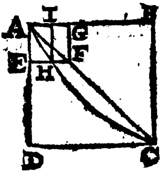


Unto a right-lined figure given ABEDC to deſcribe another figure P like and alike ſituate, which alſo ſhall be equal to another right-lined figure given F.

a Make the rectangle AL = ABEDC; *b* alſo a 45. 1.
upon BL make the rectangle BM=F; betwixt b 44. 1.
AB and BH *c* find out a mean proportional NO; c 13. 6.
Upon NO *b* make the polygone B like to the d 18. 6.
right-lined figure given ABEDC. I ſay the po-

e *cor.*20.6. lygone P ſo made ſhall be equal to F that was
f 1. 6. given.
g 14. 5. For ABEDC (AL) P :: *e* AB. BH :: *f* AL.
h *conſtr.* BM. Therefore P *g* $=$ BM. *h* $=$ F. *Which was to be done.*

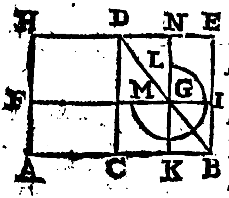
PROP. XXVI.



If from a Parallelogram ABCD, be taken away another Parallelogram AGFE, like unto the whole, and in like ſort ſet, having alſo an angle common with it EAG; then is that Parallelogram about the ſame diagonal AC with the whole.

If you deny AC to be the common diagonal, then let AHC be it, cutting EF in H, and let HI be drawn parallel to AE. Then are Pgrs. EI, DB
a 24. 6. *a* like. *b* therefore AE.EH :: AD.DC :: *c* AE.EF.
b 1. *def.*6. and *d* conſequently $EH = EF$. *f Which is abſurd.*
c *hyp.*
d 9. 5.
f 9. *ax.*

PROP. XXVII.

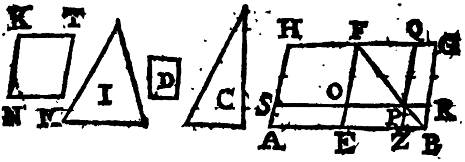


Of all parallelograms, AD, AG applied to the ſame right line AB, and wanting in figure by the parallelograms CE, KI like and alike ſet to the Pgr. AD which is deſcribed upon the half line, the greateſt is that AD which is applied to the half being like to the defect KI.

For becauſe that GE *a* $=$ GC, and KI added
a 43. 1. in common. *b* thence is KE $=$ CI *c* $=$ AM.
b 2. *ax.* add CG in common. *d* then is AG $=$ to the
c 36. 1. Gnomon MBL. But the Gnomon MBL *e* $\sqsupset$
d 2. *ax.* CE (AD.) Therefore AG $\sqsupset$ AD. *Which was*
e 9. *ax.* *to be demonſtrated.*

PROP.

PROP. XXVIII.

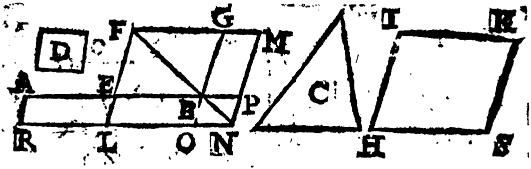


Upon a right line given AB, to apply a parallelogram AP equal to a right-lined figure given C, deficient by a parallelogram ZR which is like to another parallelogram given D. * *Now it is requiſite that the* * 27. 6.
right-lined figure given C, whereunto the Pgr. to be applied AP muſt be equal, be not greater than the Pgr. AF which is applied upon the half line, the defects being like, namely the defect of the Pgr. AF, which is applied to the half line, and the defect of the Pgr. D to be applied whoſe defect is to be like to the Pgr. given.

Biſect AB in E; upon EB *a* make the Pgr. a 18. 6.
EG like to the Pgr. D; and *b* let $EG = C + I$. b *ſch.* 45. 1.
c Make the Pgr. $NT = I$, and like to the Pgr. c 25. 6.
given D, or EG; Draw the diameter FB; Make $FO = KN$, and $FQ = KT$; thro' O and Q draw the parallels SR, QZ. Then is the Pgr. AP that which was ſought.

For the Pgrs. D, EG, OQ, NT, ZR are all *d* d *conſtr.* &
like one to the other, and the Pgr. $EG =$ *e* NT 24. 6.
$+ C =$ *e* $OQ + C$. *f* wherefore $C =$ to the e *conſtr.*
Gnomon OBQ *g* $= AO + PG =$ *h* $AQ + EP$ f 3. *ax.*
$= AP$. *Which was to be done.* g 2. *ax.*
h 43. 1.

PROP. XXIX.

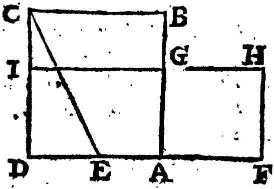


Upon a right line given AB. to apply a Pgr. AN equal to a right-lined figure given C, exceeding by a Pgr. OP, which ſhall be like to another Pgr. given D.

a 18. 6. Biſect AB in E. Upon EB *a* make a Pgr. EG
b 25. 6. like to D, which was given. And *b* let the Pgr.
$HK = EG + C$, and like to D given, or EG. Make
c 3. 1. $FEL \overset{c}{=} IH$, and *c* $FGM = IK$. Thro' L, M draw
the parallels MN and RN, and AR parallel to NM.
produce ABP, GBO. Draw the diameter FBN.
Then is AN the parallelogram required.

d *conſtr.* For the pgrs. D, HK. LM. EG are *d* like, *e* there-
e 24. 1. fore the pgr. OP is like to the pgr. LM, or D.
f *conſtr.* Alſo $LM \overset{f}{=} HK \overset{f}{=} EG + C$. *g* Therefore $C =$
g 3. *ax.* to the Gnomon ENG. But $AL \overset{h}{=} LB \overset{k}{=} BM$.
h 36. 1. *l* therefore $C = AN$. *Which was to be done.*
k 43. 1.
l 2. *and* 1. *ax.*

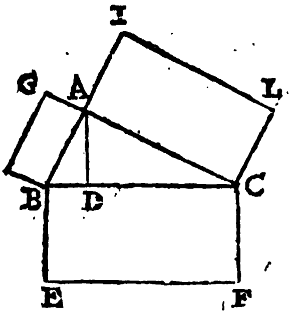
PROP. XXX.



To cut a finite right line given AB according to extreme and mean ratio (AB. AG :: AG. GB.)

a 11. 2. *a* Cut AB in G, in ſuch wiſe that $AB \times BG = AGq$. *b*
b 17. 6. Then BA. AG :: AG. GB. *Which was to be done.*

PROP. XXXI.



In right-angled triangles BAC, any figure BF deſcribed upon the ſide BC ſubtending the right angle BAC, is equal to the figures BG, AL, deſcribed upon the ſides BA, AC, containing the right angle, like and alike ſituate to the former BF.

From the right angle BAC let down a per-
pendicular AD. Becauſe that DC. CA :: *a* CA. a *cor.* 8. 6.
CB, *b* therefore AL. BF :: DC. CB. Alſo, be- b *cor.* 20. 6.
cauſe DB. BA :: *a* BA. BC, *b* therefore BG. BF
:: DB. BC. *c* therefore AL + BG. BF :: DC + c 24. 5.
DB (BC.) BC. *d* Therefore AL + BG = BF. d *ſchol.* 14. 5.
Which was to be dem.

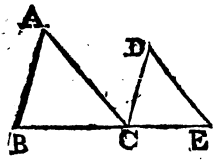
Or thus: BG. BF :: *e* BAq. BCq. And *e* AL. e 22. 6.
BF :: ACq. BCq. *f* therefore BG + AL. BF :: f 24. 5.
BAq + ACq. BCq. *g* Therefore whereas BAq + g *ſch.* 14. 5.
ACq = *h* BCq. *h* thence is BG + AL = BF. *Which* h 47. 1.
was to be dem.

Coroll.

From this Propoſition you may learn how to add or ſubſtract any like figures, by the ſame method that is uſed in adding and ſubſtracting of ſquares, in *Schol.* 47. 1.

PROP.

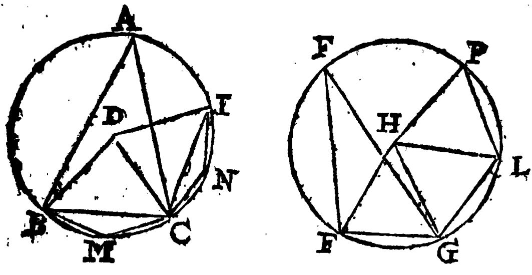
PROP. XXXII.



If two triangles ABC, DCE having two ſides proportional to two (AB. AC :: DC. DE.) be ſo compounded or ſet together at one angle ACD that their homologous ſides
be alſo parallel AB to DC, and AC to DE,) then the
a 29. 1. *remaining ſides of thoſe triangles (BC, CE) ſhall be*
b *hyp.* *found placed in one ſtrait line.*

c 6. 6. For the angle A *a* = ACD *a* = D, and AB.
d 2. *ax.* AC *b* :: DC. DE. *c* therefore the angle B = DCE,
e 32. 1. Therefore the angle B + A *d* = ACE. But the
f 1. *ax.* angle B + A + ACB *e* = 2 right. *f* therefore the
g 14. 1. angle ACE + ACB = 2 right. *g* therefore BCE
is a right line. *Which was to be dem.*

PROP. XXXIII.



In equal circles DBCA, HFGP, the angles BDC, FHG have the ſame ratio with their peripheries BC, FG on which they inſiſt; whether the angles be ſet at the centers (as BDC, FHG) *or at the circumferences, A, E: And in like ſort are the ſectors BDC, FHG, becauſe deſcribed upon the centers.*

Draw the right lines BC, FG. Make CI=CB, and GL=FG=LP, and join DI, HL, HP.

The arch BC *a*=CI. *a* alſo the arch FG, GL, LP are equal. *b* therefore the angle BDC=CDI, *b* and the angle FGH=GHL=LHP. Therefore the arch BI is as multiple of the arch BC, as the angle BDI is of the angle BDC. And in like manner is the arch EP as multiple of the arch FG, as the angle FHP is of the angle FHG. But if the arch BI $\sqsubset$, =, $\sqsupset$ FP. *c* then likewiſe is the angle BDI $\sqsubset$, =, $\sqsupset$ FHP. Therefore is the arch BC.FG :: *d* the angle BDC.FHG *e* :: $\frac{BDC}{2}.\frac{FHG}{2}$ *f* :: A. E. *Which was to be dem-*

a 28. 3.
b 27. 3.
c 27. 3.
d 6. *def.* 5.
e 15. 5.
f 20. 3.

Moreover, the angle BMC *g* = CNI; *h* and therefore the ſegment BCM = CIN. *k* Alſo the triangle BDC = CDI, *l* wherefore the ſector BDCM = CDIN. After the ſame manner are the ſectors FHG, GHL, LHP equal one to the other. Therefore ſince accordingly as the arch BI $\sqsubset$, =, $\sqsupset$ FGP, ſo is likewiſe the ſector BDI $\sqsubset$, =, $\sqsupset$ FHP; *m* thence ſhall be the ſector BDC. FHG :: the arch BC. FG. *Which was to be demonſtrated.*

g 27. 3.
h 24. 3.
k 4. 1.
l 2. *ax.*
m 6. *def.* 5

Coroll.

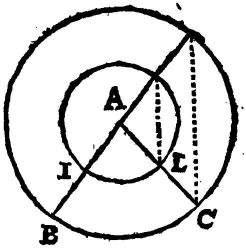
1. Hence, *As ſector is to ſector, ſo is angle to angle.* 11. 5.

2. *The angle BDC in the center is to four right angles, as the arch BC, on which it inſiſts, to the whole circumference.*

For as the angle BDC is to a right angle, ſo is the arch BC to a quadrant. Therefore BDC is ſo to four right angles as the arch BC is to four quadrants, that is, the whole circumference. Alſo the angle A. 2 right :: the arch BC periphery.

3. Hence, *The arches IL, BC of unequal circles which ſubtend equal angles, whether at the centers, as IAL and BAC, or at the periphery are like.*

For IL. periph. :: angle IAL (BAC) 4 right. Alſo Arch BC. periph :: angle BAC. 4 right. Therefore IL. periph :: BC. periph. And conſequently the arches IL and BC are like. Whence



4. *Two ſemidiameters AB, AC cut off like arches IL, BC from concentrical peripheries.*

The End of the ſixth Book.

THE SEVENTH BOOK OF EUCLIDE's ELEMENTS.

Definitions.

1. UNity is that, by which every thing that is, is called One.

II. Number is a multitude compoſed of units.

III. One number is a part of another, the leſſer of the greater, when the leſſer meaſures the greater.

Every part is denominated from that number by which it meaſures the number whereof it is a part; as 4 is called the third part of 12, becauſe it meaſures 12 by 3.

IV. But the leſſer number is termed Parts, when it meaſures not the greater.

All parts whatſoever are denominated from thoſe two numbers, by which the greateſt common meaſure of the two numbers meaſures each of them; as 10 is ſaid to be $\frac{2}{3}$ *of the number 15; becauſe the greateſt common meaſure, which is 5, meaſures 10 by 2, and 15 by 3.*

V. A number is Multiple (or Manifold) a greater in compariſon of a leſſer, when the leſſer meaſures the greater.

VI. An even number is that which may be divided into two equal parts.

VII. But an odd number is that which cannot be divided into two equal parts; or, that which differs from an even number by an unit.

VIII. A number evenly even is that which an even number meaſures by an even number.

IX. But a number evenly odd is that which an even number meaſures by an odd number.

X. A number oddly odd is that which an odd number meaſures by an odd number.

XI. A prime (or firſt) number is that, which is meaſured only by an unit.

XII. Numbers prime the one to the other, are ſuch as only an unit doth meaſure, being their common meaſure.

XIII. A compoſed number is that which ſome certain number meaſures.

XIV. Numbers compoſed the one to the other, are they, which ſome number, being a common meaſure to them both, does meaſure.

In this, and the preceding definition, unity is not a number.

XV. One number is ſaid to multiply another, when the number multiplied is ſo often added to it ſelf, as there are units in the number multiplying, and another number is produced.

Hence in every multiplication a unit is to the multiplier, as the multiplicand is to the product.

Obſ. *That many times, when any number are to be multiplied as (A into B) the conjunction of the letters denotes the product: So* $AB = A \times B$, *and* $CDE = C \times D \times E$.

XVI. When two numbers multiplying themſelves produce another, the number produced is called a plain number; and the numbers which multiplied one another, are called the ſides of that number: *So* 2 (C) × 3 (D) = 6 = CD *is a plain number.*

XVII. But when three numbers multiplying one another produce any number, the number produced is termed a ſolid number; and the numbers multiplying one another, are the ſides thereof: *So* 2 (*C*) × 3 (*D*) × 5 (*E*) = 30 = *CDE* *is a ſolid number.*

XVIII. A ſquare number is that which is equally equal; or, which is contained under two equal numbers. *Let A be the ſide of a ſquare; the ſquare is thus noted, AA, or Aq. or* A^2.

XIX. A Cube is that number which is equally equal equally, or, which is contained under three equal numbers. *Let A be the ſide of a Cube; the Cube is thus noted, AAA, or Ac. or* A^3.

In this definition, and the three foregoing, unity is a number.

XX. Numbers are proportional, when the firſt is as multiple of the ſecond, as the third is of the fourth; or, the ſame part; or, when a part of the firſt number meaſures the ſecond, and the ſame part of the third meaſures the fourth, equally: and on the contrary. *So A. B :: C. D, that is,* 3.9 :: 5.15.

XXI. Like plane, and ſolid numbers are they, which have their ſides proportional: *Namely, not all the ſides, but ſome.*

XXII. A perfect number is that which is equal to all its aliquot parts.

As 6 *and* 28. *But a number that is leſs than its aliquotparts is called an* Abounding *number; and a greater a* Diminutive: *ſo* 12 *is an Abounding,* 15 *a Diminutive number.*

XXIII. One number is ſaid to meaſure another, by that number, which, when it multiplies, or is multiplied by it, it produces.

In Diviſion, a unit is to the quotient as the diviſor is to the dividend. Note, *that a number placed under another with a line betwixt them, ſignifies Diviſion:* So $\frac{A}{B}$ = A divided by B, and $\frac{CA}{B}$ = C × A divided by B.

Two numbers are called Terms or Roots of Proportion, leſſer than which cannot be found in the ſame proportion.

Poſtulates or Petitions.

1. That numbers equal or manifold to any number may be taken at pleaſure.

2. That a greater number may be taken than any number whatſoever.

3. That Addition, Subſtraction, Multiplication, Diviſion, and the Extractions of roots or ſides of ſquare and cube numbers, be alſo granted as poſſible.

Axiomes.

1. WHatſoever agrees with one or many equal numbers, agrees likewiſe with the reſt.

2. Thoſe parts that are the ſame to the ſame part, or parts, are the ſame amongſt themſelves.

3. Numbers that are the ſame parts of equal numbers, or of the ſame number, are equal amongſt themſelves.

4. Thoſe numbers, of whom the ſame number, or equal numbers, are the ſame parts, are equal amongſt themſelves.

5. Unity meaſures every number by the units that are in it; that is, by the ſame number.

6. Every number meaſures it ſelf by a unity.

7. If one number, multiplying another, produce a third, the multiplier ſhall meaſure the product by the multiplicand; and the multiplicand ſhall meaſure the ſame by the multiplier.

Hence, *No prime number is either a plane, ſolid, ſquare, or cube number.*

8. If one number meaſures another, that number by which it meaſures ſhall meaſure the ſame by the units that are in the number meaſuring, that is by the number itſelf that meaſures.

9. If a number meaſuring another, multiply that by which it meaſures, or be multiplied by it, it produces that number which it meaſures.

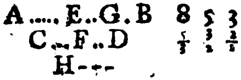
10. How many numbers ſoever any number meaſures, it likewiſe meaſures the number compoſed of them.

11. If a number meaſure any number, it alſo meaſures every number which the ſaid number meaſures.

12. A number that meaſures the whole and a part taken away doth alſo meaſure the reſidue.

PROP.

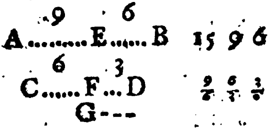
PROP. I.



If two unequal numbers AB, CD, being given, the leſſer CD be continually taken from the greater AB (and the reſidue EB from CD, &c.) by an alternate ſubſtraction, and the number remaining do never meaſure the precedent, till the unity GB be taken; then are the numbers which were given AB, CD, prime the one to the other.

If you deny it, let AB, CD have a common meaſure, namely, the number H. Therefore H meaſuring CD, does *a* alſo meaſure AE; and a 11.*ax*.7. conſequently the remainder EB; *a* therefore it likewiſe meaſures CF, and *b* ſo the remainder b 12.*ax*.7. FD; *a* wherefore it alſo meaſures EG. But it meaſured the whole EB, and *b* therefore it muſt meaſure that which remains GB, a unity, it ſelf being a number. *c Which is abſurd.* c 9.*ax*. 7.

PROP. II.



Two numbers AB, CD being given, not prime the one to the other to find out their greateſt common meaſure FD.

Take the leſſer number CD from the greater AB as often as you can. If nothing remains, *a* a 6.*ax*.7. it is manifeſt that CD is the greateſt common meaſure. But if there remains ſomething (as EB) then take it out of CD, and the reſidue FD out of EB, and ſo forward till ſome number (FD) b 1. 7. meaſure the ſaid EB (*b* for this will be before you come to a unity.) FD ſhall be the greateſt common meaſure.

For FD *c* meaſures EB, and *d* therefore alſo CF; c *conſtr.* and *e* conſequently the whole CD; *d* therefore like- d 11.*ax*.7. wiſe AE; and ſo meaſures the whole AB; Wherefore e 12.*ax*.7.

it is evident that FD is a common meaſure. If
you deny it to be the greateſt, let there be a
d 11.*ax*.7. greater (G;) then whereas G meaſures CD, it *d*
e 12.*ax*.7. muſt likewiſe meaſure AE, *e* and the reſidue EB,
g *ſuppoſ*. *d* as alſo CF, *e* and by conſequence the reſidue
h 9. *ax*.1. FD, *g* the greater the leſs. *h Which is abſurd.*

Coroll.

Hence, A numcer that meaſures two numbers, does alſo meaſure their greateſt common meaſure.

PROP. III.

A..........12
B.......8
D....4
C......6
E..2
F---

Three numbers being given, A, B, C, not prime to one another, to find out their greateſt common meaſure E.

Find out D the greateſt common meaſure of the two numbers A, B. If D meaſures C the third, it is clear that D is the greateſt common meaſure of all the three numbers. If D does not meaſure C, at leaſt D and C will be compoſed the one to the other, by the Coroll of the Prop. preceding. Therefore let E be the greateſt common meaſure of the ſaid numbers D and C, and it appears to be the number required.

a *conſtr.* For E *a* meaſures C and D, and D meaſures
b 11.*ax*.7. A and B; therefore *b* E meaſures each of the
numbers A, B, C; neither ſhall any greater (F)
c *cor*. 1. 7. meaſure them; for if you affirm that, *c* then
F meaſuring A and B, does likewiſe meaſure D
their greateſt common meaſure; and in like
manner, F meaſuring D and C, does alſo mea-
d *ſuppoſ*. ſure E *c* their greateſt common meaſure, *d* the
e 9. *ax*. 1. greater the leſs. *e Which is abſurd.*

Coroll.

Hence, A number that meaſures three numbers, does alſo meaſure their greateſt common meaſure.

PROP. IV.

A......6
B......7
B..................18
B.........9

Every leſs number A is of every greater B either a part or parts.

If A and B be prime to one another, *a* A ſhall be as many parts of the number B, as there are unities in A (as $6 = \frac{6}{7} 7$.) But if A meaſures B, it is *b* plain that A is a part of B (as $6 = \frac{1}{3} 18$.) Laſtly, if A and B be otherwiſe compoſed to one another, *c* the greateſt common meaſure ſhall determine how many parts A does contain of B; as $6 = \frac{2}{3} 9$.

a 4. *def.* 7.
b 3. *def.* 7.
c 4. *def.* 7.

PROP. V.

A......6 D....4
6 6 4 4
B......G......C 12. E....H....F 8.

If a number A be a part of a number BC, and another number D the ſame part of another number EF; then both the numbers together (A + D) ſhall be the ſame part of both the numbers together (BC + EF,) which one number A is of one number BC.

For if BC be reſolved into its parts BG, GC, equal to A; and EF alſo into its parts EH, HF equal to D; *a* the number of parts in BC ſhall be equal to the number of parts in EF. Therefore ſince A + D *b* = BG + EH = GC + HF, thence A + D ſhall be as often in BC + EF, as A is in BC. *Which was to be dem.*

a *hyp.*
b *conſt.* and 2. *ax.* 1.

Or thus. Let $a = \frac{x}{2}$, and $b = \frac{y}{2}$, then $2.a = x$, *c* and $2 b = y$. wherefore $2 a + 2 b = x + y$. therefore $a + b = \frac{x + y}{2}$. *Which was to be dem.*

c 2. *ax.* 1.

PROP. VI.

3 3 4 4
A...G...B6 D....H.....E8
C.........9 F...........12

If a number AB be parts of a number C, and another number DE the ſame parts of another number F; then both numbers together AB + DE ſhall be of both numbers together C + F the ſame parts, that one number AB is of one number C.

Divide AB into its parts AG, GB; and DE into its parts DH, HE. The multitude of parts in
both AB, DE is equal by ſuppoſition; wherefore
a *hyp.* ſince AG *a* is the ſame part of the number C,
that DH is of the number F; therefore AG +
b 5. 7. DH *b* ſhall be the ſame part of the compounded
number C + F, that one number AG is of one
number C. *b* In like manner GB+HE is the ſame
part of the ſaid C + F, that one number GB is
c 2. *ax.* 7. of one number C. *c* Therefore AB + DE is the
ſame parts of C + F, that AB is of C. *Which was to be dem.*

Or thus. Let $a = \frac{2}{3}x$, and $b = \frac{2}{3}y$, and $x + y = g$, then, becauſe $3a = 2x$, and $3b = 2y$, is $3a + 3b = 2x + 2y = 2g$. therefore $a + b = \frac{2}{3}g = \frac{2}{3}:x+y$.

PROP. VII.

5 3
A.....E...B8
6 10 6
G......C..........F......D16

If a number AB be the ſame part of a number CD, that a part taken away AE is of a part taken away CF; then ſhall the reſidue EB be the ſame part of the the reſidue FD, that the whole AB is of the whole CD.

a 1. *poſt.* 7. *a* Let EB be the ſame part of the number GC that
b 5. 7. AB is of CD, or AE of CF. *b* Therefore AE+EB
is the ſame part of CF+GC that AE is of CF, or
c 6. *ax.* 1. AB of CD. *c* therefore GF = CD. Take away
d 3. *ax.* 1. CF common to both, and *d* there remains GC =
e 2. *ax.* 7. FD. *e* Wherefore EB is the ſame part of the reſidue FD (GC) that the whole AB is of the whole CB. *Which was to be dem.*

Or thus. Let $a + b = x$; and $c + d = y$; and $x = 3\,y$, in like manner as $a = 3\,c$; I ſay $b = 3\,d$. For $3\,c + 3\,d$ *f* $= 3\,y = x$ *g* $= a + b$. take away from both $3\,c$ *g* $= a$, and *h* there remains $3\,d = b$. *Which was to be dem.* f 1. 2. g *hyp.*

PROP. VIII.

6 2 4 2 2
A......H..G....E..L..B 16
18 6
C..................F......D 24

If a number AB be the ſame parts of a number CD, that a part taken away AE is of a part taken away CF; the reſidue alſo EB ſhall be of the reſidue FD the ſame parts, that the whole AB is of the whole CD.

Divide AB into AG, GB, parts of the number CD; alſo AE into AH, HE, parts of the number CF; and take $GL = AH = HE$. *a* wherefore $HG = EL$. And becauſe *b* $AG = GB$, *c* therefore $HG = LB$. Now whereas the whole AG is the ſame part of the whole CD that the part taken away AH is of the part taken away CF, *d* the reſidue HG or EL ſhall be the ſame part alſo of the reſidue FD that AG is of CD. In like manner, becauſe GB is the ſame part of the whole CD, that HE or GL are of CF, *d* therefore the reſidue LB ſhall be the ſame part of the reſidue FD that GB is of the whole CD. Therefore $EL + LB$ (EB) is the ſame parts of the reſidue FD, that the whole AB is of the whole CD. *Which was to be dem.*
a 3. *ax.* 1.
b *conſtr.*
c 3. *ax.* 1.
d 7. 7.

Or thus more eaſily. Let $a + b = x$, and $c + d = y$. Alſo $y = \frac{2}{3}\,x$ as well as $c = \frac{2}{3}\,a$; or, *e* which is the ſame, $3\,y = 2\,x$; and $3\,c = 2\,a$. I ſay $d = \frac{2}{3}\,b$. For $3\,c + 3\,d$ *f* $= 3\,y = 2\,x$ *f* $= 2\,a + 2\,b$. *g* Therefore $3\,c + 3\,d = 2 + a\,2\,b$. take away from each $3\,c$ *h* $= 2\,a$. and *k* there remains $3\,d = 2\,b$. *l* therefore $d = \frac{2}{3}\,b$. *Which was to be dem.*
e 9. *ax.* 7.
f 1. 2.
g 1. *ax.* 1.
h *hyp.*
k 3. *ax.* 1.
l 8. *ax.*

PROP. IX.

A....4
4 4
B...G....C8
5 D.....5
5 5
E..... H..... F 10

If a number A be a part of a number BC, and another number D the ſame part of another number EF; then alternately what part or parts the firſt A is of the third D, the ſame part or parts ſhall the ſecond BC be of the fourth EF.

A is ſuppoſed ⊐ D. therefore let BG, GC, and EH, HF, parts of the numbers BC, EF be equal; BG and GC to A; and EH, HF to D. The multitude of parts is put equal in both. But it is clear that
a 1.ax. 7. GB is *a* the ſame part or parts of EH, that GC is of
and 4. 7. HF; *b* wherefore BC (BG+GC) is the ſame part
b 5.&6.7. or parts of EF (EH+HF) that BG alone (A) is of EH alone (D.) *Which was to be dem.*

Or thus. Let $a = \frac{b}{3}$, and $c = \frac{d}{3}$; or $3a = b$ and
*15. 5. $\frac{3c}{a} = d$. then is $\frac{c}{a}{}^{*} = \frac{3c}{3a} \overset{*}{=} \frac{d}{b}$.

PROP. X.

A..G..B4
C......6
5 5
D.....H.....E10
F...........15

If a number AB be parts of a number C, and another number DE the ſame parts of another number F; then alternately, what part or parts the firſt AB is of the third DE, the ſame parts or part ſhall the ſecond C be of the fourth F.

AB is taken ⊐ DE, and C⊐F. Let AG, GB, and DH, HE be parts of the numbers C and F, *viz.* as many in AB, as in DE. It is manifeſt that
a 9.7. AG is the ſame part of C, that DH is of F. *a* whence alternately AG is of DH, and likewiſe GB
b 5.&9.7. of HE, and *b* ſo conjointly AB of DE the ſame part, or parts, that C is of F. *Which was to be dem.*

Or thus, Let $a = \frac{2b}{3}$, & $c = \frac{2d}{3}$. or $3a = 2b$, & $3c = 2d$, Then is $\frac{c}{a} = \frac{3c}{3a} = \frac{2d}{2b} = \frac{d}{b}$.

PRO[illegible]

PROP. XI.

4 3
A....E...B 7
8 6
C........F......D 14

If a part taken away AB be to a part taken away CF, as the whole AB is to the whole CD, the reſidue alſo EB ſhall be to the reſidue FD, as the whole AB is to the whole CD.

Firſt, let AB be ⊐ CD; *a* then AB is either a part or parts of the number CD; and likewiſe AE is *b* the ſame part or parts of CF; *c* theref. the reſidue EB is the ſame part or parts of the reſidue FD, that the whole AB is of the whole CD. *b* and ſo AB. CD :: EB. FD. But if AB be ⊏ CD, then according to what is already ſhewn, will CD. AB :: FD. EB. therefore by inverſion AB. CD :: EB. FD.

a 4. 7.
b 20. *def.* 7.
c 7. *or.* 8. 7.

PROP. XII.

A, 4. C, 2. E, 3.
B, 8. D, 4, F, 6.

If there be divers numbers, how many ſoever, proportional (A. B :: C. D :: E. F;) then as one of the antecedents A is to one of the conſequents B, ſo ſhall all the antecedents (A+C+E) be to all the conſequents (B+D+F.)

Firſt, let A, C, E, be ⊐ B, D, F; then (becauſe of the ſame proportions) *a* ſhall A be the ſame part or parts of B that C is of D; *b* and likewiſe conjointly A+C ſhall be the ſame part or parts of B+D that A alone is of B alone. In like manner A+C+E is the ſame part or parts of B+D+F that A is of B. *c* Therefore A+C+E. B+D+F :: A. B. But if A, C, E, be put greater than B, D, F, the ſame may be ſhewn by inverſion.

a 20. *def.* 7.
b 5, & 6, 7
c 20. *def.* 7.

PROP. XIII.

A, 3. C, 4. *If there be four numbers proportional*
B, 5. D, 12. *(A.B :: C. D.) then alternately they ſhall alſo be proportional, (A.C :: B.D.)*

First let A and C be ⊐ B and D, and A ⊐
a 20.def.7. C. By reaſon of the ſame proportion *a* ſhall A
b 9. & 10. be the ſame part or parts of B, that C is of D.
7. *b* Therefore alternately A is the ſame part or
parts of C that B is of D. and ſo A. C :: B. D.
But if A be ⊏ C, and A and C ſuppoſed ⊐ B
and D, the matter will be the ſame by inverting the proportions.

PROP. XIV.

A, 9. D, 6. *If there be numbers, how many ſoever,*
B, 6. E, 4. *A, B, C, and as many more equal to*
C, 3. F, 2. *them in multitude, which may be compared two and two in the ſame proportion (A.B :: D. E. and B.C :: E.F;) they ſhall alſo, of equality, be in the ſame proportion (A, C :: D. F.)*

a 13.7. For becauſe A. B :: D. E. *a* therefore alternately is A. D :: B. E :: *a* C. F. *a* and ſo again by changing A. C :: D. F. *Which was to be dem.*

PROP. XV.

1, D.,
B... 3 . E...... 6. *If a unit meaſure any number B, and another number D do equally meaſure ſome other number E; alternately alſo ſhall a unit meaſure the third number D, as often as the ſecond B does the fourth E.*

For ſeeing 1 is the ſame part of B, that D is
a 9.7. of E; *a* therefore alternately ſhall 1 be the ſame part of D, that B is of E. *Which was to be demonſtrated.*

PROP.

PROP. XVI.

B, 4. A, 3.
A. 3. B, 4.
AB, 12. BA, 12.

If two numbers, A, B, multiplying themſelves the one into the other, produce any numbers AB, BA; the numbers produced AB and BA ſhall be equal the one to the other.

For becauſe $AB = A \times B$, *a* therefore ſhall 1 be as often in A, as B in AB, *b* and by conſequence alternately 1 ſhall be as often in B as A in AB. But for that $BA = B \times A$, *a* therefore ſhall 1 be as often in B, as A in BA. therefore as often as 1 is in AB, ſo often is 1 in BA. and *c* ſo $AB = BA$. *Which was to be dem.*

a 15.*def.*7.
b 15.7.
c 4.*ax.*7.

PROP. XVII.

A, 3.
B, 2. C, 4.
AB, 6. AC, 12.

If a number A multiplying two numbers B, C, produce other numbers, AB, AC; the numbers produc'd of them ſhall be in the ſame proportion that the numbers multiplied are. (AB. AC :: B. C.)

For being $AB = A \times B$, *a* therefore ſhall 1 be as often in A, as B in AB. *a* Likewiſe becauſe $AC = A \times C$, therefore ſhall 1 be as often in A, as C in AC. and ſo alſo B as often in AB as C in AC. *b* wherefore B. AB :: C. AC. *c* and therefore alſo alternately B. C :: AB. AC. *Which was to be dem.*

a 15.*def.*7.
b 20.*def.*7.
c 13.7.

PROP. XVIII.

C, 5. C, 5.
A, 3. B, 9.
AC, 15. BC, 45.

If two numbers A, B, multiplying any number C, produce other numbers AC, BC; the numbers produced of them ſhall be in the ſame proportion that the numbers multiplying are. (A. B :: AC. BC.)

For *a* $AC = CA$, and BC *a* $= CB$; ſo the ſame C multiplying A and B produces AC and BC. *b* therefore A. B :: AC. BC. *Which was to be dem.*

a 16.7.
b 17.7.

Schol.

Schol.

Hence is deduced the vulgar manner of reducing fractions ($\frac{3}{5}$, $\frac{7}{9}$) to the ſame denomination. For multiply 9 both by 3 and 5, and they produce $\frac{27}{45} = \frac{3}{5}$; becauſe by this, 3. 5 :: 27. 45. Likewiſe multiply 5 by 7 and 9; there ariſes $\frac{35}{45} = \frac{7}{9}$; becauſe 7. 9 :: 35. 45.

PROP. XIX.

A, 4. B, 6. C, 8. D, 12.
AD, 48. BC, 48.

If there be four numbers in proportion (A. B :: C. D) the number produced of the firſt and fourth (AD) is equal to the number which is produced of the ſecond and third (BC.) And if the number which is produced of the firſt and fourth (AD) be equal to that produced of the ſecond and third (BC) thoſe four numbers ſhall be in proportion (A. B :: C. D.)

a 17. 7.
b *hyp.*
c 18. 7.
d 9. 5.
e *hyp.*
f 7. 5.
g 17. 7.
h 18. 7.
k 11. 5.

1. *Hyp.* For AC. AD *a* :: C. D *b* :: A. B *c* :: AC. BC. *d* therefore AD = BC. *Which was to be dem.*

2. *Hyp.* Becauſe *e* AD = BC, therefore AC. AD *f* :: AC. BC. But AC. AD *g* :: C. D. and AC. BC *h* :: A. B. *k* therefore C. D :: A. B. *Which was to be demonſtrated.*

PROP. XX.

A. B. C.
4. 6. 9.
AC, 36. BB, 36.
D, 6.

If there be three numbers in proportion (A. B :: B. C) the number contained under the extremes (AC) is equal to the ſquare made of the middle (BB.) And if the number contained under the extremes be equal to that (Bq.) produced of the middle, thoſe three numbers ſhall be in proportion $\left(\frac{A}{B} \cdot \frac{B}{C} \cdot\right)$

a 1. *ax.* 7.
b 19. 7.

1. *Hyp.* For take D = B. *a* therefore A. B :: D (B.) C. *b* wherefore AC = BD, *a* or BB. *Which was to be demonſtrated.*

2. *Hyp.* Becauſe AC *c* = BD, *d* therefore A. c *hyp.*
B :: D (B) C. *Which was to be dem.* d 19. 7.

PROP. XXI.

A ... G .. B 5. E 10
C .. H . D 3. F 6

Numbers AB, CD, being the leaſt of all that have the ſame proportion with them (E, F) do equally meaſure the numbers, E, F, having the ſame proportion with them; the greater AB the greater E, and the leſſer CD the leſſer F.

For AB. CD *a* :: E. F. *b* therefore alternately a *hyp.*
AB. E :: CD. F. *c* therefore AB is the ſame part b 13. 7.
or parts of E that CD is of F. Not parts; for if c 20. *def.* 7.
ſo, then let AG, GB be parts of the number E;
and CH, HD, parts of the number F. *e* therefore
AG. E :: CH. F. and by inverſion AG. CH *d* :: d 13. 7.
E. F *e* :: AB. CD. therefore AB, CD are not the e *hyp.*
leaſt in their proportion; which is contrary to
the Hypotheſis. Therefore, *&c.*

PROP. XXII.

A, 4. D, 12.
B, 3. E, 8.
C, 2. F, 6.

If there be three numbers A, B, C; and other numbers equal to them in multitude, D, E, F; which may be compared two and two in the ſame proportion: And if alſo the proportion of them be perturbed (A. B :: E. F. and B. C :: D. E.) then of equality they ſhall be in the ſame proportion (A. C :: D. F.)

For becauſe A. B *a* :: E. F, therefore ſhall AF a *hyp.*
= BE; and becauſe B. C :: *a* D. E, *b* therefore b 19. 7.
BE = CD. *c* and conſequently AF = CD. *d* Where- c 1. *ax.* 1.
fore A. C :: D. F. *Which was to be dem.* d 19. 7.

PROP.

PROP. XXIII.

A, 9. B, 4.
C----D---
E--

Numbers prime the one to the other, A, B, are the leaſt of all numbers that have the ſame proportion with them.

If it be poſſible, let C and D be leſs than A
a 21.7. and B, and in the ſame proportion; *a* therefore
C meaſures A equally as D meaſures B, namely,
b 23.*def*.7. by the ſame number F; and ſo C ſhall be *b* as
c 15.7. often in A as 1 is in E; *c* likewiſe alternately, E
as often in A as 1 in C. By the like inference as
many times as 1 is in D, ſo many times ſhall E
be in E. Therefore E meaſures both A and B;
which conſequently are not prime the one to the
other, *contrary to the Hypotheſis.*

PROP. XXIV.

A, 9. B, 4.
C---
D--- E--

Numbers A, B, being the leaſt of all that have the ſame proportion with them, are prime the one to the others.

If it be poſſible, let A and B have a common
meaſure C; and let the ſame meaſure A by D,
a 9.*ax*.7. and B by E; *a* therefore CD=A, *b* and CE=B.
b 17.7. *b* Wherefore A.B :: D.E. But D and E are leſſer
than A and B, as being but parts of them.
Therefore A and B are not the leaſt in their proportion, *againſt the Hypotheſis.*

PROP. XXV.

A, 9. B, 4.
C, 3. D--

If two numbers A, B, be prime the one to the other, the number C meaſuring one of them A, ſhall be prime to the other number B.

For if you affirm any other D to meaſure the
numbers B and C, *a* then D meaſuring C does
a 11.*ax*.7. alſo meaſure A; and conſequently A and B are
not prime the one to the other. *Which is againſt the Hypotheſis.*

PROP. XXVI.

A, 5. C, 8.
B, 3.
AB, 15. E - - - -
F - - - -

If two numbers, A, B, be prime to any number C, the number alſo produced of them A, B ſhall be prime to the ſame C.

If it be poſſible, let the number E be a common meaſure to AB, and C; and let $\frac{AB}{E}$ be $=$ F; *a* a 9. *ax.* 7.
thence AB $=$ EF; *b* wherefore alſo E. A :: B. F. b 19. 7.
But becauſe A is prime to C, which E meaſures, c 25. 7.
c therefore E and A are prime to one another, *d* d 23. 7.
and ſo leaſt in their own proportion, *e* and con- e 21. 7.
ſequently they muſt meaſure B and F; namely
F ſhall meaſure B, and A ſhall meaſure F.
Therefore ſeeing E meaſures both B and C, they
ſhall not be prime to one another: *Contrary
to the Hypotheſis.*

PROP. XXVII.

A, 4. B, 5.
Aq, 16.
D, 4.

If two numbers A, B, be prime to one another, that alſo which is produc'd of one of them (Aq) ſhall be prime to the other B.

Take D $=$ A; therefore each of D, and A a 1. *ax.* 7.
are prime to B. *b* wherefore AD or Aq is prime b 26. 7.
to B. *Which was to be dem.*

PROP. XXVIII.

A, 5. C, 4.
B, 3. D, 2.
AB, 15. CD, 8.

If two numbers A, B, be prime to two numbers C, D, each to either of both, the numbers alſo produced of them AB, CD ſhall be prime to one another.

For being A and B are prime to C, *a* therefore a 26. 7.
ſhall AB alſo be prime to the ſame. And by the

ſame

b 26. 7. ſame reaſon ſhall AB be prime to D. *b* Therefore AB is prime to CD. *Which was to be dem.*

PROP. XXIX.

A, 3. B, 2.
Aq, 9. Bq, 4.
Ac, 27. Bc, 8.

If two numbers A, B, be prime to one another, and each multiplying himſelf produce another number (Aq, and Bq;) *then the numbers produced of them* (Aq, Bq) *ſhall be prime to one another. And if the numbers given at firſt, A, B, multiplying the ſaid produced numbers*(Aq, Bq)*produce others* (Ac, Bc) *thoſe numbers alſo ſhall be prime to one another: And this ſhall ever happen about the extremes.*

a 27. 7. For becauſe A is prime to B, *a* therefore Aq ſhall be prime to B. and Aq being prime to B, *a* therefore Aq ſhall alſo be prime to Bq. Again, becauſe A is as well prime to B and Bq. as Aq
b 28. 7. is to the ſaid B and Bq, *b* therefore ſhall A x Aq, that is, Ac, be prime to B x Bq, that is, to Bc: And ſo forth of the reſt.

PROP. XXX.

8 5
A........B..... C 13 D----

If two numbers AB, BC, be prime the one to the other, then both added together (AC) ſhall be prime to either of them AB, BC. And if both added together AC be prime to any one of them AB, the numbers alſo given in the beginning AB, BC ſhall be prime to one another.

1. *Hyp.* For if you would have AC, AB to be
a 12. ax. 7. compoſed, let D be the common meaſure: *a* this ſhall meaſure the reſidue BC: and therefore AB, BC, are not prime to one another; *which is againſt the Hypotheſis.*

2. *Hyp.* AC, AB being taken for prime to one another, let D be the common meaſure of AB,
b 10. ax. 7. BC. *b* But ſeeing that meaſures the whole AC, therefore AC, AB, are not prime to one another; *contrary to the Hypotheſis.*

Coroll.

Hence, A number which being compounded of two, is prime to one of them, is alſo prime to the other.

PROP. XXXI.

A, 5. B, 8. *Every prime number A is prime to every number B, which it meaſures not.*

For if any common meaſure does meaſure both A, B, *a* then A will not be a prime number; *contrary to the Hypotheſis.* a 11. *def.* 7.

PROP. XXXII.

A, 4. D, 3.
B, 6. E, 8.
AB, 24.

If two numbers A, B, multiplying one another produce another AB, and ſome prime number D meaſure the number produced of them AB; then ſhall it alſo meaſure one of thoſe numbers, A or B, which were given at the beginning.

Suppoſe the number D not to meaſure the number A, and let $\frac{AB}{D}$=E. *a* then AB=DE, *b* whence D. A :: B. E. *c* But D is prime to A; *d* therefore D and A are the leaſt in their proportion; *e* and conſequently D meaſures B as often as A meaſures E. *Which was to be dem.* a 9. *ax.* 7. b 19. 7. c *hyp. and* 31. 7. d 23. 7. e 21. 7.

PROP. XXXIII.

A, 12. B, 2. *Every compoſed number A is meaſured by ſome prime number B.*

Let one or more numbers *a* meaſure A, of which let the leaſt be B; that ſhall be a prime number: a 13. *def.* 7. For if it be ſaid to be compoſed, then ſome *a* leſſer number ſhall meaſure it, *b* which ſhall alſo conſequently meaſure A. Wherefore B is not the leaſt of them which meaſure A, *contrary to the Hyp.* b 11. *ax.* 7.

PROP. XXXIV.

A, 9. *Every number A, is either a prime, or meaſured by ſome prime number.*

For A is either neceſſarily a prime or a compoſed number. If it be a prime, 'tis that we
a 33. 7. affirm. If compoſed, *a* then ſome prime number meaſures it. *Which was to be dem.*

PROP. XXXV.

A, 6. B, 4. C, 8.
D, 2.
E, 3. F, 2. G, 4.

H -- I -- K ----
L ---

How many numbers ſoever A, B, C, being given, to find the leaſt numbers E, F, G, that have the ſame proportion with them.

a 23. 7. If A, B, C, be prime to one another, *a* they
b 3. 7. ſhall be the leaſt in their proportion. If they be compoſed, *b* let their greateſt common meaſure be D, which let meaſure them by E, F, G. Theſe are then leaſt in the proportion A, B. C.
c 9. ax. 7. For D × E, F, G, *c* produces ABC, *d* therefore
d 17. 7. they are all in the ſame proportion. But allow other numbers H, I, K to be the leaſt in the
e 21. 7. ſame proportion; *e* which ſhall therefore equally
f 9. ax. 7. meaſure ABC, namely by the number L. *f* there-
g 1. ax. 1. fore L × H,I,K, ſhall produce A,B,C, *g* and con-
h 19. 7. ſequently ED = A = HL. *h* from whence E. H ::
k *ſuppoſ.* L. D. But E *k* ⊏ K; *l* therefore L ⊏ D, and ſo
l 20. *def.* 7. D is not the greateſt common meaſure of A, B, C. *Which is againſt the Hypotheſis.*

Coroll.

Hence, The greateſt common meaſure of how many numbers ſoever, does meaſure them by the numbers which are leaſt of all that have the ſame proportion with them. Whereby appears the vulgar method of reducing fractions to the leaſt terms.

PROP.

PROP. XXXVI.

Two numbers being given A, B, to find out the leaſt number which they meaſure.

A, 5. B, 4.
AB, 20.
D------
E---F---

1. *Caſe.* If A and B be prime the one to the other, AB is the number required. For it is manifeſt that A and B meaſure AB. If it be poſſible, let A and B meaſure
ſome other number $D \sqsupset AB$, if you pleaſe by
E, and F. *a* therefore $AE = D = BF$, *b* and ſo A. a 9. *ax*. 7.
B :: F. E. But becauſe A and B *c* are prime the & 1. *ax*. 1.
one to the other, *d* and ſo leaſt in their propor- b 19. 7.
tion, A ſhall *e* equally meaſure F as B does E. c *hyp.*
But B. E *f* :: AB. AE (D) *g* Therefore AB ſhall d 23. 7.
alſo meaſure D, which is leſs than it ſelf. *Which* e 21. 7.
is abſurd. f 17. 7.
g 20. *def*. 7.

A, 6. B, 4. F----
C, 3. D, 2. G--H---
AD, 12.

2. *Caſe.* But if A and B be compoſed one to
another, *h* let there be h 35. 7.
found C and D the
leaſt in the ſame proportion. *k* therefore $AD = BC$; k 19. 7.
and AD or BC ſhall be the number ſought for.

For it is *l* plain that B and D do meaſure AD l 7. *ax*. 7.
or BC. Conceive A and B to meaſure $F \sqsupset AD$,
namely A by G, and B by H. *m* therefore $AG =$ m 9. *ax*. 7.
$F = BH$. *n* whence A. B :: H. G *o* :: C. D. *p* and n 19. 7.
conſequently C equally meaſures H as D does G. o *conſtr.*
But D. G *q* :: AD. AG (F.) therefore AD *r* mea- p 21. 7.
ſures F, the greater the leſs. *Which is abſurd.* q 17. 7.
r 20. *def*. 7.

Coroll.

Hence, If two numbers multiply the leaſt that are in the ſame proportion, the greater the leſs, and the leſs the greater, the leaſt number which they meaſure ſhall be produced.

PROP. XXXVII.

A, 2. B, 3.
E 6
C ----F----D

If two numbers A, B, meaſure any number CD, the leaſt number which they meaſure E ſhall alſo meaſure the ſame CD.

If you deny it, take E from CD as often as you can, and leave FD ⊐ E. therefore ſeeing A and B *a* meaſure E, *b* and E meaſures CF, *c* likewiſe A and B will meaſure CF. But *a* they meaſure the whole CD; *d* therefore alſo they meaſure the reſidue FD; and conſequently E is not the leaſt which A and B meaſure; *contrary to the Hypotheſis*.

a *hyp.*
b *conſtr.*
c 11.*ax.*7.
d 12.*ax.*7.

PROP. XXXVIII.

A, 3. B, 4. C, 6.
D, 12.

Three numbers being given A, B, C, to find out the leaſt which they meaſure.

a 36. 7.

a Find D to be the leaſt that two of them A and B do meaſure; which if the third C do alſo meaſure, it is manifeſt that D is the number ſought for. But if C do not meaſure D, let E be the leaſt that C and D do meaſure. E ſhall be the number required.

A, 2. B, 3. C, 4.
D, 6. E, 12.
F ---

For it appears by the 11. *ax.* 7. that A, B, C meaſure E; and it is eaſily ſhewn that they meaſure no other leſs than F. For if you affirm they do, *b* then D meaſures F, *b* and conſequently E meaſures the ſame F, the greater the leſs. *Which is abſurd.*

b 37. 7.

Coroll.

Hence it appears, that if three numbers meaſure any number, the leaſt alſo, which they meaſure, ſhall meaſure the ſame.

PROP.

PROP. XXXIX.

A, 12.
B, 4. C, 3.

If any number B meaſure a number A, the number meaſured A, ſhall have a part C denominated of the number meaſuring B.

For becauſe $\frac{A}{B}$ *a* $=$ C, *b* ſhall A $=$ BC. *c* therefore $\frac{A}{C} = B$. *Which was to be dem.* a *hyp.* b 9. *ax.* 7. c 7. *ax.* 7.

PROP. XL.

A, 15.
B, 3. C, 5.

If a number A have any part whatſoever B, the number C, from which the part B is denominated, ſhall meaſure the ſame.

For being BC *a* $=$ A, *b* thence $\frac{A}{C} = B$. *Which was to be dem.* a *hyp.* & 9. *ax.* 7. b 7. *ax.* 7.

PROP. XLI.

$\frac{1}{2}$ G, 12.
$\frac{1}{3}$ H - - -
$\frac{1}{4}$

To find out a number G, which being the leaſt, contains the parts given $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$.

a Let G be found the leaſt which the denominators, 2, 3, 4, meaſure; *b* it is evident that G has the parts $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$. If it be poſſible let H $\sqsubset$ G have the ſame parts; *c* therefore 2, 3, 4, meaſure H; and ſo G is not the leaſt which 2, 3, 4 meaſure: *againſt the conſtr.* a 38. 7. b 39. 7. c 40. 7.

The End of the ſeventh Book.

THE EIGHTH BOOK OF EUCLIDE's ELEMENTS.

PROP. I.

A, 8. B, 12. C, 18. D, 27.
E - F - - G - - - H - - - -

IF there be divers numbers how many ſoever in continual proportion, A,B,C,D, and their extremes A,D, prime to one another; then thoſe numbers A,B,C,D, are the leaſt of all numbers that have the ſame proportion with them.

For, if it be poſſible, let there be as many others E, F, G, H, leſs than A, B, C, D, and in the ſame proportion with them. *a* Therefore of
a 14.7. equality A.D :: E.H. and conſequently A and
b 23.7. D are prime numbers, *b* and ſo the leaſt in their proportion, *c* equally meaſuring E and H, which
c 21.7. are leſs than themſelves. *Which is abſurd.*

PROP. II.

1.
A, 2. B, 3.
Aq, 4. AB, 6. Bq, 9.
Ac, 8. AqB, 12. ABq, 18. Bc, 27.

To find out the leaſt numbers continually proportional, as many as ſhall be required, in the proportion given of A to B.

Let A and B be the leaſt in the proportion given; then Aq, AB, Bq, ſhall be three laſt in the ſame continual proportion that A is to B.

For AA. AB *a* :: A. B *a* :: AB. BB. Likewiſe a 17. 7,
becauſe A and B are *b* prime to one another, *c* b 24. 7.
ſhall Aq, Bq, be alſo prime to one another, *d* c 29. 7.
and ſo Aq, AB, Bq, are ∺ the leaſt in the pro- d 1. 8.
portion of A to B.

Moreover, I ſay Ac, AqB, ABq, Bc, are the
four leaſt in the proportion of A to B. For AqA.
AqB *e* :: A. B. *e* :: ABA (AqB.) ABB. *e* and A. e 17. 7.
B :: ABq. BBq (Bc.) Therefore ſince Ac, and Bc, f 29. 7.
are *f* prime to one another, likewiſe *g* ſhall Ac, g 1. 8.
AqB, ABq, Bc be the four leaſt ∺ in the proportion of A to B. In the ſame manner may you find out as many proportional numbers as you pleaſe. *Which was to be done.*

Coroll.

1. Hence, If three numbers, being the leaſt, are proportional, their extremes ſhall be ſquares; if four, cubes.

2. How many extremes proportional ſoever there be, being by this prop. found to be the leaſt in the given proportion, they are prime to one another.

3. Two numbers, being the leaſt in the given proportion, do meaſure all the mean numbers whatſoever of the leaſt in the ſame proportion; becauſe they ariſe from the multiplication of them into certain other numbers.

4. Hence alſo it appears by the conſtruction, that the ſeries of numbers 1, A, Aq, Ac; 1, B, Bq. Bc; Ac, AqB, ABq, Bc conſiſts of an equal multitude of numbers; and conſequently, the extreme numbers of how many ſoever the leaſt continually proportionals are the laſt of as many other continually proportionals from a unite, as the extreme Ac, Bc, of the continually proportionals Ac, AqB. ABq, Bc, are the leaſt of as many proportionals from a unite, 1, A, Aq, Ac; and 1, B, Bq, Bc.

5. 1, A, Aq, Ac; and B, BA, BAq; and Bq, ABq, are ∺ in the proportion of 1 to A. Alſo B, Bq, Bc; and A, AB, ABq; and Aq, AqB are ∺ in the proportion of 1 to B.

PROP. III.

A, 8. B, 12. C, 18. D, 28.

If there be numbers continually proportional, how many ſoever, A,B,C,D, being alſo the leaſt of all that have the ſame proportion with them; their extremes A, D, are prime to one another.

a 2. 8. For if there be *a* found as many numbers the leaſt in the proportion of A to B, they ſhall be no other than A, B, C, D; therefore, by the ſecond *Coroll.* of the precedent *Prop.* the extremes A and D are prime to one another. *Which was to be dem.*

PROP. IV.

A, 6. B, 5. C, 4. D, 3.
H, 4. F, 24. E, 20. G, 15.
I-- K--- L---

Proportions how many ſoever being given, in the leaſt numbers (A to B, and C to D;) to find out the leaſt numbers continually proportional in the proportions given.

a 36. 7. *a* Find out E the leaſt number which B and C
b 3. *poſt*. 7. do meaſure; and let B meaſure E *b* as often as A
does another F, *viz.* by the ſame number H. *b*
Alſo let C meaſure the ſaid E as often as D meaſures
another G. then F, E, G, ſhall be the leaſt
c 9. *ax.* 7. in the proportions given. For AH *c* = F, and
d 18. 7. BC *c* = E; *d* therefore A. B :: AH. BH *c* :: F. E.
e 7. 5. In like manner C. D :: E. G; therefore F, E, G,
are continually proportional in the proportions
given. And they are moreover the leaſt in the
ſaid proportions: For conceive other numbers,
f 21. 7. I,K,L, to be the leaſt; *f* then A and B muſt equally
meaſure I and K, *f* and C and D likewiſe
K and L; and ſo B and C meaſure the ſame K.
g 37. 7. *g* Wherefore alſo E meaſures the ſame number
K, which is leſs than it ſelf. *Which is abſurd.*

A, 6. B, 5. C, 4. D, 3. E, 5. F, 7.
H, 24. G, 20. I, 15. K, 21.

But three proportions being given, A to B, C to D, and E to F; find out as before three numbers H, G, I, the leaſt continually in the proportions of A to B, and C to D. Then if E meaſures I, *h* take another number K which may be equally meaſur'd by F; and thoſe four numbers H, G, I, K, ſhall be continually the leaſt in the given proportions; which we need go no other way to prove than we did in the firſt part. h 3.*poſt*.7.

A, 6. B, 5. C, 4. D, 3. E, 2. F, 7.
H, 24. G, 29. I, 25.
M, 48. L, 40. K, 30. N, 105.

If E do not meaſure I, let K be the leaſt which E and I do meaſure; and as often as I meaſures K, let G as often meaſure L, and H alſo M. ſo likewiſe let F meaſure N as often as E meaſures K. The four numbers, M, L, K, N ſhall be leaſt, continually in the given proportions; which we may demonſtrate as before.

PROP. V.

C, 4. E, 3.
D, 6. F, 16. ED, 18.
CD, 24. EF, 48.

Plane numbers CD, EF, are in that proportion to one another which is compoſed of their ſides.

For becauſe CD.ED *a* :: C.E; *a* and ED.EF :: D.F. & $\frac{CD}{EF}$ *b* $= \frac{CD}{ED} + \frac{D}{E}$ *c* then ſhall be the proportion $\frac{CD}{EF} = \frac{C}{E} + \frac{D}{F}$. *Which was to be dem.*

a 17.7.
b 20.*def*.5
c 11.5.

PROP. VI.

A, 16. B, 24. C, 36. D, 54. E, 81.
F, 4. G, 6. H, 9.

If there be numbers continually proportional how many ſoever, A, B, C, D, E, and the firſt A do not meaſure the ſecond B, neither ſhall any of the other meaſure any one of the reſt.

a 20.*def*.7. Becauſe A does not meaſure B, *a* neither ſhall
any one meaſure that which next follows; being
b 35. 7. A.B :: B.C :: C.D.&c. *b* Take three numbers,
F, G, H, the leaſt in the proportion of A to B.
therefore ſince A does not meaſure B, *a* neither
c 5. *ax*. 7. ſhall F meaſure G. *c* therefore F is not a unit. But
d 3. 3. F and H are prime to one another; and ſo, *d* being
e 14. 7. of equality A. C :: F.H. and F does not meaſure
H, *a* neither ſhall A meaſure C; and conſequently
neither ſhall B meaſure D, nor C meaſure
E, &c. becauſe A. C *e* :: B. D *e* :: C.E, &c.
In like manner four or five numbers being taken
the leaſt in the proportion of A to B, it will appear
that A does not meaſure D and E; nor does
B meaſure E and F, &c. Wherefore none of them
ſhall meaſure any other. *Which was to be dem.*

PROP. VII.

A, 3. B, 6. C, 12. D, 24. E, 48.

If there be numbers continually proportional how many ſoever A, B, C, D, E, and the firſt A meaſure the laſt E, it ſhall alſo meaſure the ſecond B.

a 6. 7. If you deny that A meaſures B, *a* then neither
ſhall it meaſure E; *Which is contrary to the Hyp.*

PROP. VIII.

A,24. C,36. D,54. B,81.
G,8. H,12. I,18. K,27.
E,32. L,48. M,72. F,108.

If between two numbers A, B, there fall mean proportional numbers in continual proportion C, D; as many mean continually proportional numbers as fall between them, ſo many alſo mean continually proportional numbers ſhall fall between two other numbers E, F, which have the ſame proportion with them. (L, M.)

a 35. 7. *a* Take G,H,I,K, the leaſt ∺ in the proportion
b 14. 7. of A to C; *b* of equality ſhall G.K :: A.B *c* :: E.F.
c *hyp*. But G, and K *d* are prime to one another. *e*
d 3. 8. Wherefore G meaſures E as often as K does F.
e 21. 7. Let H meaſure L, and I likewiſe M by the ſame
f *conſtr*. number. *f* therefore E, L, M, F, are in ſuch proportion
as G,H,I,K, that is as A,B,C,D. *Which was to be dem.*

PROP. IX.

1.
E,2. F,3.
G,4. H,6. I,9.
A,8. C,12. D,18. B,17.

If two numbers A, B be prime to one another, and mean numbers in continual proportion C,D fall between them; as many mean numbers in continual proportion as fall between them, ſo many means alſo (E, G; and F. I) *ſhall fall in continual proportion between either of them and a unity.*

It is evident that 1, E, G, A, and 1, F, I, B, are ∺, and as many as A, C, D, B, namely by the 4. Coroll. 2. 8. *Which was to be dem.*

PROP. X.

A,8. I,12. K,18. B,27.
E,4. DF,6. G,9.
D,2. F,3.
1.

If between two numbers A,B, and a unit, numbers continually proportional (E, D, and F, G,) *do fall, how many mean numbers in continual proportion fall between either of them and a unit, ſo many means alſo ſhall fall in continual proportion between them, I, K.*

For E, DF, G, and A, DqF (I) DG, (K,) B, are ∺ by 2. 8. therefore, &c.

PROP. XI.

A,2. B,3.
Aq,4. AB,6. Bq,9.

Between two ſquare numbers Aq, Bq, there is one mean proportional number AB: and the ſquare Aq to the ſquare Bq is in double proportion of that of the ſide A to the ſide B.

a It is manifeſt that Aq,AB,Bq, are ∺; *b* and a 17. 7.
conſequently alſo $\frac{Aq}{Bq} = \frac{A}{B}$ doubly. *W.W. to be dem.* b 10. *def.* 7.

PROP. XII.

Ac,27. AqB,36, ABq,48. Bc,64.
A, 3. B, 4.
Aq, 9. AB, 12. Bq, 16.

Between two cube numbers, Ac, Bc, there are two mean proportional numbers AqB, ABq: and the cube Ac is to the cube Bc in treble proportion of that in which the ſide A is to the ſide B.

a 2. 8. *a* For Ac, AqB, ABq, Bc, are ∺ in the propor-
b 10 def. 5. tion of A to B; *b* and therefore $\frac{Ac}{Bc} = \frac{A}{B}$ trebly.

Which was to be dem.

PROP. XIII.

A, 2. B, 4. C, 8.
Aq,4. AB,8. Bq,16. BC,32. Cq,64.
Ac,8. AqB,16. ABq,32. Bc,64. BqC,128. BCq,256.
(Cc, 512.

If there be numbers in continual proportion how many ſoever A, B, C; and every of them multiplying it ſelf produce certain numbers; the numbers produced of them Aq, Bq, Cq, ſhall be proportional: And if the number firſt given A, B, C, multiplying their products Aq, Bq, Cq, produce other numbers, Ac, Bc, Cc, they alſo ſhall be proportional; and this ſhall ever happen to the extremes.

a 2. 8. For Aq, AB, Bq, BC, Cq *a* are ∺ *b* therefore
b 14. 7. of equality Aq.Bq :: Bq.Cq. *Which was to be dem.*

a Alſo Ac, AqB, ABq, Bc, BqC, BCq, Cc, are ∺; *b* therefore likewiſe of equality Ac.Bc :: Bc. Cc. *Which was to be dem.*

PROP. XIV.

Aq,4. AB,12. Bq,36.
A, 2. B, 6.

If a ſquare number Aq meaſure a ſquare number Bq, the ſide alſo of the one (A) ſhall meaſure the ſide of the other (B:) and if the ſide of one ſquare A meaſure the ſide of another B, the ſquare Aq ſhall likewiſe meaſure the ſquare Bq.

1. *Hyp.* For Aq.AB *a* :: AB. Bq. therefore ſee- a 2.&11.8
ing by the Hypotheſis Aq meaſures Bq, *b* it b 7.8.
ſhall meaſure alſo AB. But Aq. AB :: A. B. *c* c 20.*def*.7.
therefore A meaſures B. *Which was to be dem.*

2 *Hyp.* A meaſures B. *c* therefore Aq ſhall as
well meaſure AB, *c* as AB meaſures Bq; *d* and con- d 11.*ax*.7.
ſequently Aq meaſures Bq. *Which was to be dem.*

PROP. XV.

A, 2. B, 6. *If a cube num-*
Ac,8.AqB,24.ABq,72.Bc,216. *ber Ac meaſures a*
cube number Bc,
then the ſide of the one (A) ſhall meaſure the ſide of the other (B:) And if the ſide A of one cube Ac meaſure the ſide B of the other Bc, alſo the cube Ac ſhall meaſure the cube Bc.

1. *Hyp.* For Ac, AqB, ABq, Bc *a* are ∺, there- a 2.&12.8
fore Ac, *b* meaſuring the extreme Bc. ſhall alſo b *hyp.*
c meaſure the ſecond AqB. But Ac. Aq. B :: A. c 7.8.
B. *d* therefore A ſhall alſo meaſure B. d 20.*def*.7.

2. *Hyp.* A meaſures B; *d* therefore Ac mea-
ſures AqB, which alſo meaſures ABq, and that e 11.*ax*.7.
Bc; *e* therefore Ac ſhall meaſure Bc. *Which was to be dem.*

PROP. XVI.

Aq, 4. B, 9. *If a ſquare number Aq do not mea-*
Aq,16.Bq,81. *ſure a ſquare number Bq, neither ſhall*
the ſide of the one A meaſure the ſide of the other B: And if A the ſide of the one ſquare Aq do not meaſure B the ſide of the other Bq, neither ſhall the ſquare Aq meaſure the ſquare Bq.

1. *Hyp.* For if you affirm that A meaſures B, *a* a 14.8.
then Aq alſo ſhall meaſure Bq. *againſt the Hyp.*

2. *Hyp.* If you maintain Aq to meaſure Bq; *a* then likewiſe A ſhall meaſure B. *contrary to the Hypotheſis.*

PROP. XVII.

A,2. B,3.
Ac,8. Bc,27.

If a cube number Ac do not meaſure a cube number Bc, neither ſhall the ſide of one A, meaſure the ſide of the other B: And if A the ſide of one cube Ac do not meaſure B the ſide of the other Bc, neither ſhall the cube Ac meaſure the cube Bc.

a 15. 8. 1. *Hyp.* Let A meaſure B; *a* then Ac ſhall meaſure Bc. *againſt the Hyp.*

2. *Hyp.* Let Ac meaſure Bc; then A ſhall meaſure B; *which is alſo againſt the Hypotheſis.*

PROP. XVIII.

C,6. D,2.
CD,12.
E,9. F,3. DE,18.
EF,27.

Between two like plane numbers CD and EF there is one mean proportional number DE: And the plane CD is to the plane EF in double proportion of that which the ſide C has to the homologous ſide (or of like proportion) E.

* 21.*def*.7. Being * by the Hypoth. C. D :: E. F. therefore by inverſion C. E :: D. F. But C.E *a* :: CD.
a 17. 7. DE; *a* and D.F :: DE.EF. *b* therefore CD.DE ::
b 11. 5. DE. EF. *c* Wherefore the proportion of CD to
c 10.*def*.5. EF is double to that of CD to DE, that is, to the proportion of C to E, or D to F.

Coroll.

Hence it is apparent, that between two like plane numbers there falls one mean proportional in the proportion of the homologous ſides.

PROP.

PROP. XIX.

CDE,30. DEF,60. FGE,120. FGH,240.
CD,6. DF,12. FG,24.
C,2. D,3. E,5. F,3. G,6. H,10.

Between two like ſolid numbers CDE, FGH, there are two mean proportional numbers DFE, FGE. And the ſolid CDE is to the ſolid FGH, in treble proportion of that which the homologous ſide C has to the homologous ſide F.

Whereas by the * hyp. C.D :: F.G, & D.E :: * 21.*def.*7.
G. H. therefore *a* by inverſion ſhall C. F :: D. a 13. 7.
G :: *a* E.H. But CD. DF *b* :: C. F, and DF. FG b 17. 7.
b :: D. G; *c* wherefore CD. DF :: DF. FG :: c 11. 5.
E. H. *d* and accordingly CDE. DFE :: DFE. d 17. 7.
FGE :: E. H :: FGE. FGH. Therefore between
CDE. FGH, fall two mean proportionals DFE,
FGE. *e* And ſo it is plain that the proportion e 10.*def.*5.
of CDE to FGH is treble to that of CDE to
FDE, or C to F. *Which was to be dem.*

Coroll.

Hereby it is manifeſt, that between two like ſolid numbers there fall two mean proportionals in the proportion of the homologous ſides.

PROP. XX.

A,12, C,18. B,27. *If between two numbers*
D,2. E,3. F,6. G,9. *A, B, there fall one mean*
proportional number C; theſe
numbers A, B, are like plane numbers.

a Take D and E the leaſt in the proportion of A a 35. 7.
to C, or C to B. then D meaſures A equally as E
does C, *viz.* by the ſame number F; *b* alſo D equally b 21. 7.
meaſures C, as E does B, *viz.* by the ſame number
G. *c* Therefore DF = A, and EG = B. *d* and con- c 9. *ax.* 7.
ſequently A and B are plane numbers. But becauſe d 16.*def.*7.
EF *c* = C *c* = DG, *e* ſhall D. E :: F. G. and alter- e 19. 7.
nately

f 21. *def.* 7. nately D.F ∷ E.G. *f* Therefore the plane numbers A and B are alſo like. *Which was to be dem.*

PROP. XXI.

A, 16. C, 24. D, 36. B, 54.
E, 4. F, 6. G, 9.
H, 2. P, 2. M, 4. K, 3. L, 3. N, 6.

If between two numbers A, B there fall two mean proportional numbers C, D; thoſe numbers A, B are like ſolid numbers.

a 2. 8. *a* Take E, F, G, the leaſt ∺ in the proportion
b 10. 8. of A to C. *b* then E and G are like plane numbers: let the ſides of this be H and P, and of that
c 21. *def.* 7 K and L. *c* therefore H.K ∷ P.L ∷ *d* E. F. But E,
d *cor.* 18. 8. F, G, do *e* equally meaſure A, C, D, *viz.* by the
e 21. 7. ſame number M. and likewiſe the ſaid numbers E, F, G, do equally meaſure the numbers C, B, D, *viz.* by the ſame number N. *f* Therefore A = EM
f 9. *ax.* 7. = HPM, *f* and B = GN = KLN; *g* and ſo A and
g 17. *def.* 7. B are ſolid numbers. But for that C *f* = FM, and
h 17. 7. D *f* = FN, therefore ſhall M. N *h* ∷ FM. FN *k*
k 7. 5. ∷ C.D *l* ∷ E.F ∷ H.K ∷ P. L. *m* wherefore A and
l *conſtr.* B are like ſolid numbers. *Which was to be dem.*
m 21. *def.* 7.

Lemma.

AE, BF, CG, DH,
A, B, C, D,
E, F, G, H.

If proportional numbers A, B, C, D, meaſure proportional numbers AE, BF, CG, DH, by the numbers E, F, G, H, theſe numbers (E, F, G, H,) ſhall be proportional.

a 19. 7. For being AEDH *a* = BFCG, *a* and AD = BC,
b 1. *ax.* 7. *b* thence will $\frac{AEDH}{AD} = \frac{BFCG}{BC}$ *c* that is, EH = FG.
c 9. *ax.* 7. *a* Therefore E.F ∷ G.H. *Which was to be dem.*

Coroll.

d 15. *def.* 7. Hence $\frac{Bq}{Aq} = \frac{B}{A} \times \frac{B}{A}$ *d* For 1.B ∷ B.Bq *d* and 1.A ∷
e *lem. prec.* A.Aq. *e* theref. $\frac{1}{A}$. B ∷ $\frac{B}{A}$ $\frac{Bq}{Aq}$ *d* theref. $\frac{Bq}{Aq} = \frac{B}{A} \times \frac{B}{A}$

In like manner $\frac{B}{Ac} \times \frac{B}{Ac} = \frac{Bc}{Acc}$, and ſo of the reſt.

PROP.

PROP. XXII.

Aq, B, C. *If three numbers Aq, B, C be*
4. 8. 16. *continually proportional, and the firſt Aq a ſquare, the third C ſhall alſo be a ſquare.*

For becauſe AqC *a*= Bq, *b* thence is $C = \frac{Bq}{Aq}$ *c*$= Q.\frac{B}{A}$; But it is plain that $\frac{B}{A}$ is a number, *d* becauſe $\frac{Bq}{Aq}$ or C is a number. Theref. if three, *&c.*

a 20. 7.
b *ax.* 7.
c *cor. of the lem. prec.*
d *hyp. and* 14. 8.

PROP. XXIII.

Ac, B, C, D. *If four numbers Ac, B, C, D,*
8, 12, 18, 27. *be continually proportional, and the firſt of them Ac a cube, the fourth alſo D ſhall be a cube.*

For becauſe AcD *a*= BC, *b* therefore $D = \frac{BC}{Ac}$ *c*$= \frac{B}{Ac} \times C$; that is (becauſe AcC = *d* Bq, and *b* thence $C = \frac{Bq}{Ac}$ $D = \frac{B}{Ac} \times \frac{Bq}{Ac}$ *e*$= \frac{Bc}{Acc}$ *c* $= C \times \frac{B}{Ac}$. But it is evident *c* that $\frac{B}{Ac}$ is a number, becauſe $\frac{Bc}{Acc}$ or D is ſuppoſed a number. Therefore if four numbers, *&c.*

a 19. 7.
b 7. *ax.* 7.
c *cor. of the prec. lem.*
d 20. 7.
e 15. 8.

PROP. XXIV.

A, 16. 24. B, 36. *If two numbers A, B, be in the*
C, 4. 6. D, 9. *ſame proportion one to another, that a ſquare number C is to a ſquare number D, and the firſt A be a ſquare number, the ſecond alſo B ſhall be a ſquare number.*

Between C and D being ſquare numbers, * and ſo between A and B having the ſame proportion, *a* falls one mean proportional. Therefore *b* being

* 8. 8.
a 11. 8.
b *hyp.*

A

c 12. 8. A is a ſquare number, c B alſo ſhall be a ſquare number. *Which was to be dem.*

Coroll.

1. Hence, If there be two like numbers AB, CD (A.B :: C.D) and the firſt AB be a ſquare, the ſecond alſo CD ſhall be a ſquare.

*12.& 18 8. * For AB. CD :: Aq. Cq.

2. From hence it appears, that the proportion of any ſquare number to any other not ſquare, cannot poſſibly be declared in two ſquare numbers. Whence it cannot be Q. Q :: 1. 2. nor 1. 3 :: Q. Q, &c.

PROP. XXV.

C,64. 96. 144. D,216.
A,8. 12. 18. B,27.

If two numbers A,B, be in the ſame proportion one to another, that a cube number C is to a cube number D, the firſt of them A being a cube number; the ſecond B ſhall likewiſe be a cube number.

a 12. 8.
b 8. 8.
c *hyp.*
d 23. 8.

a Between the cube numbers C and D, *b* and ſo between A and B having the ſame proportion, fall two mean proportionals; therefore *c* becauſe A is a cube, *d* ſhall B be a cube alſo. *Which was to be dem.*

Coroll.

1. Hence, If there be two numbers ABC, DEF (A. B :: D. E, and B. C :: E. F;) and the firſt ABC be a cube, the ſecond DEF ſhall be a cube alſo.

*12,&19, 8. * For ABC. DEF :: Ac. Dc.

1. It is perſpicuous from hence, that the proportion of any cube number to any other number not a cube cannot be found in two cube numbers.

PROP. XXVI.

A,20. C,30. B,45.
D,4. E,6. F,9.

Like plane numbers A, B, are in the ſame proportion one to another, that a ſquare number is in to a ſquare number.

Between A and B *a* falls one mean propor- a 18.8.
tional number C; *b* take three numbers D,E,F, b 2.8.
the leaſt ∺ in the proportion of A to C. the
extremes D, F, *b* ſhall be ſquare numbers. But
of equality A. B *c* :: D.F. therefore A.B :: Q.Q. c 14.7.
Which was to be dem.

PROP. XXVII.

A,16. C,24. D,36. B,54.
E,8. F,12. G,18. H,27.

Like ſolid numbers A, B, are in the ſame proportion one to another, that a cube number is in to a cube number.

a Between A and B fall two mean proportio- a 19.8.
nal numbers, namely C and D: *b* take four b 2.8.
numbers E, F, G, H the leaſt ∺ in the propor-
tion of A to C; *b* the extremes E, H, are cube
numbers. But A. B *c* :: E. H :: C. C. *Which* c 14.7.
was to be dem.

Schol.

1. From hence is inferred, that no numbers in See *Cla-*
proportion ſuperparticular, ſuperbipartient, or *vius.*
double, or any other manifold proportion not
denominated from a ſquare number, are like
plane numbers.

2. Likewiſe, that neither any two prime numbers, nor any two numbers prime one to another, not being ſquares, can be like plane numbers.

The End of the eighth Book.

THE NINTH BOOK OF EUCLIDE's ELEMENTS.

PROP. I.

A, 6. B, 54.
Aq, 36. 108. AB, 324.

IF two like plane numbers A, B multiplying one another, produce a number AB, the number produced AB ſhall be a ſquare number.

a 17.7. For A. B *a* :: Aq. AB; wherefore ſince
b 18.8. one mean proportional *b* falls between A
c 8.8. and B, *c* likewiſe one mean proportional number
ſhall fall between Aq and AB: therefore being
d 22.8. the firſt Aq is a ſquare number, *d* the third AB
ſhall be a ſquare number too. *Which was to be demonſtrated.*

Or thus. Let ab, cd, be like plane numbers;
x 19.7. namely a. b :: c.d. *x* therefore ad = bc. and ſo like-
y 1. *ax.* 7. wiſe ab cd, or ad bc, = ad ad = Q: ad.

PROP. II.

A, 6. B, 54.
Aq, 36. AB, 324.

If two numbers A, B, multiplying one another, produce a ſquare number AB, thoſe numbers A, B are like plane numbers.

a 17.7. For A. B *a* :: Aq. AB; wherefore being be-
b 11.8. tween Aq, AB, *b* there falls one mean propor-
c 8.8. tional number, *c* likewiſe one mean ſhall fall be-
d 20.8. tween A and B. *d* therefore A and B are like
planes. *Which was to be dem.*

PROP. III.

A, 2. Ac, 8. Acc, 64. *If a cube number Ac multiplying it ſelf produce a number Acc. the number produced Acc ſhall be a cube number.*

For 1. A *a* :: A. Aq *b* :: Aq. Ac. therefore be- a 15 *def.* 7.
tween 1 and Ac fall two mean proportionals. But b 17. 7.
1. Ac *a* :: Ac. Acc. *c* therefore between Ac and c 8. 8.
Acc, fall alſo two mean proportionals: and ſo d 13. 8.
by conſequence ſeeing Ac is a cube, *d* Acc ſhall
be a cube alſo. *Which was to be dem.*

Or thus; aaa (Ac) multiplied into itſelf makes aaaaaa (Acc;) this is a cube, whoſe ſide is aa.

PROP. IV.

Ac, 8. Bc, 27. Acc, 64. AcBc, 216. *If a cube number Ac multiplying a cube number Bc, produce a number AcBc, the produced number AcBc ſhall be a cube.*

For Ac. Bc *a* :: Acc. AcBc. But between Ac a 17. 7.
and Bc *b* two mean proportional numbers fall; b 12. 8.
c therefore there fall as many between Acc and c 8. 8.
AcBc. So that whereas Acc is a cube number, *d* d 23. 8.
AcBc ſhall be ſuch alſo. *Which was to be dem.*

Or thus. AcBc = aaabbb (ababab) = C: ab.

PROP. V.

Ac, 8. B, 27. Acc, 64. AcB, 216. *If a cube number Ac multiplying a number B produce a cube number AcB, the number multiplied B ſhall alſo be a cube.*

For Acc. AcB. *a* :: Ac. B. But between Acc a 17. 7.
and AcB *b* fall two mean proportionals; *c* there- b 12. 8.
fore alſo as many ſhall fall between Ac and B. c 8. 8.
whence Ac being a cube number, *d* B ſhall be a d 23. 8.
cube number too. *Which was to be dem.*

PROP. VI.

A,8. Aq,64. Ac,512. *If a number A multiplying it ſelf produce a cube Aq, that number A it ſelf is a cube.*

a *hyp.* For becauſe Aq *a* is a cube, and AqA (Ac) *b*
b 19.*def.*7. alſo a cube; therefore *c* ſhall A be a cube. *Which*
c 5. 9. *was to be dem.*

PROP. VII.

A,6. B,11. AB,66. *If a compoſed number A*
D,2. E,3. *multiplying any number B, produce a number AB, the number produced AB ſhall be a ſolid number.*

a 13.*def.*7. Being A is a compoſed number, *a* ſome other
b 9. *def.*7. number D meaſures it, conceive by E. *b* there-
c 17.*def.*7. fore A = DE : *c* whence DEB = AB is a ſolid number. *Which was to be dem.*

PROP. VIII.

1. a, 3: a^2, 9 a^3, 27. a^4, 81. a^5, 243. a^6, 729.

If from a unit there be numbers continually proportional how many ſoever (1. a, a^2, a^3, a^4, &c.) *the third number from a unit* a^2 *is a ſquare number; and ſo are all forward, leaving one between* (a^4, a^6, a^8, &c.) *But the fourth* a^3 *is a cube number; and ſo are all forward, leaving two between* (a^6, a^9, &c.) *The ſeventh alſo* a^6, *is both a cube number and a ſquare; and ſo are all forward, leaving five between* (a^{12}, a^{18}, &c.)

For 1. a^2 = Q. a. and a^4 = aaaa = Q. aa. and a^6 = aaaaaa = Q. aaa, &c.

2. a^3 = aaa = C. a. and a^6 = aaaaaa = C. aa and aaaaaaaaa = C. aaa, &c.

3. a^6 = aaaaaa = C. aa = Q. aaa. therefore, &c.

a *hyp.* Or according to *Euclide*; Becauſe 1. a :: *a*. a^2.
b 20.7. *b* ſhall a^2 = Q: a. therefore ſeeing a^2, a^3, a^4, are ∺,
c 12.8. *c* the third a^4 ſhall be a ſquare number; and ſo likewiſe a^6, a^8, &c. Alſo becauſe 1. a *a* :: a^2. a^3.
d 23.8. therefore ſhall a^3 *b* = $a^2 \times a$ = C: a. *d* therefore the fourth from a^3, namely a^6, ſhall likewiſe be a cube, &c. and conſequently a^6 is both a cube and a ſquare number, &c.

PROP.

PROP. IX.

1. a. 4. a^2, 16. a^3, 64. a^4, 256, &c.
1. a. 8. a^2, 64. a^3, 512. a^4, 4096.

If from a unit there be numbers how many ſoever continually proportional (1. a, a^2, a^3, &c.) *and the number following the unit* (a) *be a ſquare; then all the reſt,* a^2, a^3, a^4, &c. *ſhall be ſquares too. But if the number next the unit* (a) *be a cube, then all the following numbers* a^2, a^3, a^4, &c. *ſhall be cube numbers.*

1. *Hyp.* For a^2, a^4, a^6, &c. are ſquare numbers by the prec. Prop. alſo being a is taken to be a
ſquare, *a* therefore the third a^3 ſhall be a ſquare; a 22. 8.
and likewiſe a^5, a^7, &c. and ſo all.

2. *Hyp.* a is taken to be a cube, *b* therefore a^4, b 23. 8.
a^7, a^{10} are cubes: but by the prec. a^3, a^6, a^9, c 20. 7.
&c. are cubes: laſtly, becauſe 1. a :: a. aa. *c* there- d 3. 9.
fore ſhall a^2 = Q: a. but a cube multiplied into e 23. 8.
it ſelf *d* produces a cube; therefore a^2 is a cube, *e* and conſequently the fourth from it a^5, and in like manner a^8, a^{11}, &c. are cubes, therefore all. *Which was to be dem.*

Peradventure more clearly thus. let b be the ſide of the ſquare number a, and ſo the ſeries a, a^2, a^3, a^4, &c. will be otherwiſe expreſſed, thus, bb, b^4, b^6, b^8, &c. It is evident that all theſe numbers are ſquares, and may be thus expreſſed, Q: b, Q: bb: Q. bbb, Q: bbbb, &c.

In like manner, if b be the ſide of the cube a, the ſeries may be expreſſed thus, b^3, b^6, b^9, b^{12}. &c. or C: b, C: b^2, C: b^3, C: b^4, &c.

PROP. X.

1, a, a^2, a^3, a^4, a^5, a^6,
1, 2, 4, 8, 16, 32, 64,

If from a unit there be numbers how many ſoever continually proportional (1, a, a^2, a^3, &c.) *and the number next the unit* (a) *be not a ſquare number; then is none of the reſt following a ſquare number, excepting* a^2 *the third from the unit, and ſo all forward, leaving one be-*

tween(a^4, a^6, a^8, &c.) *But if that* (a) *which is next after the unit, be not a cube number, neither is any other of the following numbers a cube, ſaving* a; *the fourth from the unit, and ſo of all forward, leaving two between*, a^6, a^9, a^{12}, &c.

1. *Hyp.* For if it be poſſible, let a^5 be a ſquare
a *hyp.* number; therefore becauſe a. a^2 *a* :: a^4, a^5, and
b *ſuppoſ.* & by inverſion a^5. a4 :: a^2. a; and alſo a^5 and a4
8 9. are *b* ſquare numbers, and the firſt a^2 a ſquare,
c 24. 8. *c* therefore a ſhall be likewiſe a ſquare; *contrary to the Hypotheſis.*

d 24. 7. 2. *Hyp.* If it may be, let a4 be a cube; being
d of equality a4. a^6 :: a. a^3 and inverſely a^6. a4
e 25. 8. :: a3. a; and alſo being a^6 and a4 are cubes, and the firſt a3 a cube; *e* therefore a ſhall be a cube alſo; *againſt the Hyp.*

PROP. XI.

1. a, a^2, a3, a4, a5, a6.
1, 3, 9, 27, 81, 243, 729.

If there be numbers how many ſoever in continual proportion from a unit (1, a, a2, a3, &c.) *the leſs meaſureth the greater by ſome one of them that are amongſt the proportional numbers.*

a 5. *ax.* 7. & 20. *def.* 7. Becauſe 1. a :: a. aa. *a* therefore $\frac{aa}{a} = a = \frac{aaa}{aa}$.

b 14. 7. Alſo becauſe 1. aa *b* :: a. aaa. *a* therefore $\frac{aaa}{a} = aa = \frac{a4}{aa} = \frac{a5}{a3}$, &c. Laſtly becauſe 1. a3 *b* :: a. a4. therefore $\frac{a4}{a} = a3 = \frac{a6}{a3}$, &c.

Coroll.

Hence, If a number that meaſures any one of proportional numbers, be not one of the ſaid numbers, neither ſhall the number by which it meaſures the ſaid proportional numbers, be one of them.

PROP.

PROP. XII.

1, a, a2, a3, a4.
1,6,36,216,1296.
B. 3.

If there be numbers how many ſoever in continual proportion from a unit) (1, a, a2, a3, a4.) *whatſoever prime numbers B meaſure the laſt* a4, *the ſame (B) ſhall alſo meaſure the number* (a) *which follows next after the unit.*

If you ſay B does not meaſure a, *a* then B is a 31. 7.
prime to a; *b* and alſo B is prime to a2; *c* and b 27. 7.
ſo conſequently to a4, which is ſuppoſed to c 26. 7.
meaſure. *Which is abſurd.*

Coroll,

1. Therefore every prime number that meaſures the laſt, does alſo meaſure all thoſe other numbers that precede the laſt.

2. If any number not meaſuring that next to the unit, does yet meaſure the laſt, it is a compoſed number.

3. If the number next to the unit be a prime, no other prime number ſhall meaſure the laſt.

PROP. XIII.

1, a. a2, a3, a4,
1, 5, 25, 125, 625.
H -- G -- F -- E --

If from a unit be numbers in continual proportion how many ſoever (a, a2, a3. &c.) *and that after the unit* (a) *a prime; then ſhall no other meaſure the greateſt number, but thoſe which are amongſt the ſaid proportional numbers.*

If it be poſſible, let ſome other E meaſure a4,
viz. by F. *a* then F ſhall be ſome other beſide a, a *cor.*11.9.
a2,a3. But becauſe E meaſuring a4,does not mea- b 1.*cor.*12.
ſure a, *b* therefore E ſhall be a compoſed num- 9.
ber, *c* and ſome prime number meaſure it,*d* which c 33.7.
does conſequently meaſure a4. *e* and ſo is no other d 11.*ax.*7.
than a, therefore a meaſures E. So alſo may F c 3.*cor.*12.
be ſhewn to be a compoſed number, meaſuring a4,9.

measuring a4, and ſo that a meaſures F. There-
f 9. ax. 7. fore ſeeing EF f = a4 = a x a3, g ſhall a. E :: F.
g 19. 7. a3. Conſequently, whereas a meaſures E, h like-
h 20.def.7. wiſe F ſhall equally meaſure a3, viz. by the
k cor.11.9. ſame number G. k Nor ſhall G be a. or a2. there-
fore, as before, G is a compoſed number, and a
meaſures it. Wherefore being that FG f = a3
= a2 x a g ſhall a. F :: G a2. and ſo becauſe A
meaſures F, h G ſhall equally meaſure a2. viz.
by the ſame number H, k which is not a. There-
l 23.def.7. fore being GH = a2 = aa, l thence H. a :: a.
m 20. def. 7. G. and becauſe a meaſures G (as before) m H
alſo ſhall meaſure a, which is a prime number. *Which is impoſſible.*

PROP. XIV.

A, 30.
B,2. C,3. D,5.
E--F---

If certain prime numbers B, C, D, do meaſure the leaſt number A, no other prime number E ſhall meaſure the ſame, beſides thoſe that meaſured it at firſt.

a 9. ax. 9. If it is poſſible, let $\frac{A}{E}$ be = F. *a* then A = EF.
b 32. 7. *b* therefore every of the prime numbers B, C, D, meaſures one of thoſe E, F, not E, which is taken to be a prime; therefore F which is leſs than it ſelf A; *contrary to the Hyp.*

PROP. XV.

A,9. B,12. C,16.
D, 3. E,4.

If three numbers continually proportional A, B, C, be the leaſt of all that have the ſame proportion with them; any two of them added together ſhall be prime to the third.

a 35. 7. *a* Take D and E the leaſt in the proportion of
b 2. 8. A to B; *b* then A = Dq, & *b* C = Eq, *b* and B = DE.
But

But becauſe D *c* is prime to E, *d* therefore ſhall c 2. 47.
$D + E$ be prime to both D and E. * therefore d 30. 7.
$D \times D + E$ *e* $= Dq + DE$ (*f* $A + B$) is prime to * 26. 7.
E, and ſo to C or Eq. *Which was to be dem.* e 3. 2.

g In like manner $DE + Eq$ ($B + C$) is prime f *before*
to D, and conſequently to $A = Dq$. *Which was* g 27. 7.
to be dem. h 26. 7.

Laſtly, becauſe B *h* is prime to $D + E$, it ſhall k 4. 2.
alſo be prime to the ſquare of it *k* $Dq + 2\ DE$ l 30. 7.
$+ Eq$ ($A + 2\ B + C$;) *l* wherefore the ſaid B
ſhall be prime to $A + B + C$, *l* and ſo likewiſe
to $A + C$. *Which was to be dem.*

PROP. XVI.

A, 3. B, 5. C---

If two numbers A, B, be prime to one another, the ſecond B ſhall not be to any other C, as the firſt A is to the ſecond B.

If you affirm A. B :: B. C. then whereas A
and B *a* are the leaſt in their proportion, A *b* a 23. 7.
ſhall meaſure B as many times as B does C; b 21. 7.
but A *c* meaſures itſelf alſo; therefore A and B c 6. *ax.* 7.
are not prime to one another. *againſt the Hyp.*

PROP. XVII.

A, 8. B, 12. C, 18. D, 27. E---

If there be numbers how many ſoever in continual proportion A, B, C, D, and the extremes of them A, D be prime one to another, the laſt D ſhall no the to any other number E, as the firſt A is to the ſecond B.

Suppoſe A. B :: D. E. then alternately A.
D :: B. E. therefore ſeeing A and B are the
leaſt in their proportion, A *b* ſhall meaſure a 23. 7.
B, *c* and B likewiſe C, and C the follow- b 21. 7.
ing number D, *d* and ſo A ſhall meaſure c 20. *def.* 7.
the ſaid number D. Wherefore A and D d 11. *ax.* 7.

are not prime to one another, *contrary to the Hypotheſis.*

PROP. XVIII.

A, 4. B, 6. C, 9. *Two numbers being given*
Bq, 36. *A, B, to conſider if there may be a third number found proportional to them C.*

a 9. *ax.* 7. If A meaſure Bq by any number C, *a* then
b 20. 7. AC = Bq. from whence *b* it is manifeſt that A. B :: B. C. *Which was to be dem.*

A,6. B,4. Bq,16. But if A do not meaſure Bq, there will not be any third proportional. For ſuppoſe A. B :: B. C. *a* then
c 7. *ax.* 7. AC = Bq. *c* and conſequently $\frac{Bq}{A} = C$. namely A meaſures Bq. *Which is againſt the Hypoth.*

PROP. XIX.

A, 8. B, 12. C, 18. D, 27. *Three numbers being*
BC, 216. *given A, B, C, to conſider if a fourth proportional to them D may be found.*

a 9. *ax.* 7. If A meaſures BC by any number D, *a* then
b *ax.* 19. 7. AD = BC; *b* therefore it appears that A. B :: C. D. which was required.

But if A do not meaſure BC, then there can no fourth proportional be found; which may be ſhewn as in the preceding Prop.

PROP. XX.

A, 2. B, 3. C, 5. *More prime numbers may be given*
D, 30. G---- *than any multitude whatſoever of prime numbers A, B, C, propounded.*

a 38. 7. *a* Let D be the leaſt which A, B, C, meaſure; If
b 33. 7. D+1 be a prime, the caſe is plain; if compoſed, *b* then ſome prime number, conceive G, meaſures

ſures D + 1; which is none of the three A, B,
C; For if it be, ſeeing it *c* meaſures the whole c *ſuppoſ.*
D + 1, *d* and the part taken away D, *e* it ſhall d *conſtr.*
alſo meaſure the remaining unit. *Which is abſurd.* e 12. *ax.* 7.
Therefore the propounded number of prime numbers is increaſed by D + 1; or at leaſt by G.

PROP. XXI.

5 5 3 3 2 2
A.....E.....B...F...C..G..D 20.

If even numbers, how many ſoever, AB, BC, CD, be added together, the whole AD ſhall be even.

a Take EB = $\frac{1}{2}$ AB, and FC = $\frac{1}{2}$ BC, and
GD = $\frac{1}{2}$ CD. *b* it is plain that EB + FC + a 6. *def.* 7.
GD = $\frac{1}{2}$ AD. *c* therefore AD is an even num- b 12. 7.
ber. *Which was to be dem.* c 6. *def.* 7.

PROP. XXII.

A........F.B......G.C....H.D..L.E 22
9 7 5 3

If odd numbers, how many ſoever, AB, BC, CD, DE, be added together, and their multitudes even, the whole alſo AE ſhall be even.

A unit being taken from each odd number,
there will *a* remain AF, BG, CH, DL, even
numbers, *b* and thence the number compounded a 7. *def.* 7.
of them will be even, add to them the *c* even b 21. 9.
number made of the remaining units, and the c *hyp.*
d whole AE will thereby be even. *Which was* d 21. 9.
to be dem.

PROP.

PROP. XXIII.

7 5 1
A......B.....C..E.D 15.
3

If odd numbers, how many ſoever, AB, BC, CD, be added together, and the multitude of them be odd, the whole AD ſhall be odd.

For CD one of the odd numbers being taken away, the number compounded of the others AC
a 22.9. *a* is even. Whereto add CD — 1, *b* the whole
b 21.3. AE is alſo even; wherefore the unit being re-
c 7. *def*.7. ſtored, the whole AD *c* will be odd. *Which was to be dem.*

PROP. XXIV.

4 5 1
A...B.....D.C 10.
6

If an even number AB be taken away from an even number AC, that which remains BC ſhall be even.

a 7. *def*. 7. For if BD (BC — 1) be odd, *a* BC (BD + 1)
will be even. *Which was to be dem.* But if you
b *hyp.* ſay BD is even, becauſe AB *b* is even, *c* thence
c 21. 9. AD will be ſo; *a* and conſequently AC (AD+1)
will be odd, *contrary to the Hypoth.* therefore BC is even. *Which was to be dem.*

PROP. XXV.

6 1 3
A......D.C...B 10.
7

If from an even number AB, an odd number AC be taken away, the remaining number CB ſhall be odd.

a 7. *def*. 7. For AC—1 (AD) *a* is even. *b* therefore DB
b 24. 9. is even; *c* and conſequently CB (DB — 1) is
c 7. *def*.7. odd. *Which was to be dem.*

PROP. XXVI.

4 6 1
A...C......D.B 11.

If from an odd number AB be taken away an odd number CB, that which remains AC ſhall be even.

For AB − 1 (AD) and CB − 1 (CD) *a* are a 7. *def.* 7.
even; *b* therefore AD − CD (AC) is even. b 24. 9.
Which was to be done.

PROP. XXVII.

1 4 6
A.D....C.....B 11.
5

If from an odd number AB be taken away an even number CB, the reſidue AC ſhall be odd.

For AB − 1 (DB) *a* is even, and CB is ſup- a 7. *def.* 7.
poſed to be even; *b* therefore the reſidue CD is b 24. 9.
even: *c* therefore CD + 1 (CA) is odd. *Which* c 7. *def.* 7
was to be dem.

PROP. XXVIII.

A, 3.
B, 4.
AB, 12.

If an odd number A multiplying an even number B produce a number AB, the number produced AB ſhall be even.

For AB *a* is compounded of the odd a *hyp.* and
number A taken as many times as a unit is 15. *def.* 7.
contained in B an even number. *b* Therefore AB b 21. 9.
is an even number.

Schol.

In like manner, if A be an even number, AB ſhall be an even number alſo.

PROP. XXIX.

A, 3.
B, 5.
AB, 15.

If an odd number A multiplying an odd number B, produce a number AB, the number produced AB ſhall be odd.

For AB *a* is compounded of the odd a 15. *def.* 7.
number B taken as often as a unit is included in
A likewiſe an odd number. *b* Therefore AB is b 23. 9.
an odd number. *Which was to be dem.*

Schol.

Schol.

$\frac{B,12}{A,3}$ (C,4. 1. *An odd number A meaſuring an even number B, meaſures the ſame by an even number C.*

a 9. ax. 7. For if C be affirmed to be odd, then becauſe *a*
b 29. 9. B = AC. *b* therefore B ſhall be odd, *againſt the Hypotheſis.*

$\frac{B,15}{A,3}$ (C,5. 2. *An odd number A meaſuring an odd number B, meaſures the ſame by an odd number C.*

a 28. 9. For if C be ſaid to be even, *a* then AC, or B will be even, *contrary to the Hypotheſis.*

$\frac{B,16}{A,3}$ (C,5. 3. *Every number (A and C) that meaſures an odd number B, is itſelf an odd number.*

For if either A or C be affirmed to be even, B
a 28. 9. *a* ſhall be an even number, *againſt the Hypoth.*

PROP. XXX.

$\frac{B,24}{A,3}$ (C,8. $\frac{D,12}{A,3}$ (E,4.

If an odd number A meaſure an even number B; it ſhall alſo meaſure the half of it D.

a hyp. *a* Let $\frac{B}{A}$ be = C. *b* then C is an even number.
b 1. Schol. 29. 9. Therefore let E be = ½ C, then B *c* = CA *d* = 2
c 9. ax. 7. EA *e* = 2 D. *f* therefore EA = D; *g* and conſe-
d 1. 2. e hyp. quently $\frac{D}{A}$ = E. *Which was to be dem.*
f 7. ax. 1.

PROP. XXXI.

g 7. ax. 7. A,5. B,8. C,16. D--- *If an odd number A be prime to any number B, it ſhall alſo be prime to the double thereof C.*

If it be poſſible, let ſome number D meaſure A
a 3. ſchol. 29. 9. and C, *a* then D meaſuring the odd number A
ſhall be odd it ſelf, *b* and ſo ſhall meaſure B the
b 30. 9. half of the even number C. theref. A and B are not prime one to another. *Which is againſt the Hyp.*

Coroll.

It follows from hence that an odd number which is prime to any number of double progreſſion, is alſo prime to all the numbers of that progreſſion.

PROP. XXXII.

1. A, 2. B, 4. C, 8. D, 16. *All numbers A, B, C, D, &c. in double progreſſion from the binarie are evenly even only.*

It is evident that all theſe numbers 1, A, B, C,
D, *a* are even, and *b* ∺, namely in a double pro- a 6. *def.* 7.
portion, *c* and ſo every leſs meaſures the greater b 20. *def.* 7.
by ſome one of them. *d* Wherefore all are even- c 12. 9.
ly even. But for that A is a prime number, *e* d 8. *def.* 7.
no number beſides theſe ſhall meaſure any of e 13. 9.
them. Therefore they are evenly even only.
Which was to be dem.

PROP. XXXIII.

A, 30. B, 15. *If of a number A, the half B be*
D--- E -- *odd, the ſame A is evenly odd only.*

Being an odd number B *a* mea-
ſures A by two an even number, *b* therefore B is a *hyp.*
evenly odd. If you affirm it to be evenly even, b 9. *def.* 7.
c then ſome even number D meaſures it by an c 8. *def.* 7.
even number E. whence 2 B *d* = A, *d* = DE. *e* d 9. *ax.* 7.
wherefore 2. E :: D. B. and therefore as 2 *f* mea- e 9. 7.
ſures the even number E, *g* ſo D an even num- f 6. *def.* 9.
ber meaſures B an odd. *Which is impoſſible.* g 20. *def.* 7.

PROP. XXXIV.

A, 24. *If an even number A be neither doubled from two, nor have its half part odd, it is both evenly even and evenly odd.*

It is undoubtable, that A is evenly even, becauſe the half of it is not odd. But becauſe, if A be divided into two equal parts, and its half again

7. *def.* 7. into two equal parts and ſo on, we ſhall at length light upon ſome *a* odd number (not upon the number two, becauſe A is not ſuppoſed to be doubled upward from two) which ſhall meaſure
b 1 *ſch.* 29. 9. A by an even number, ſo *b* otherwiſe A it ſelf ſhould be odd, *againſt the Hypoth.* Therefore A alſo evenly odd. *Which was to be dem.*

PROP. XXXV.

A........8.
4 8
B.... F........G 12.
C..................18.
9 6 4 8
D.........H......L....K.......N 27.

If there be numbers in continual proportion how many ſoever A, BG, C, DN, and the number FG be taken from the ſecond, and KN from the laſt, equal to the firſt A; as the exceſs of the ſecond BF is to the firſt A, ſo ſhall the exceſs of the laſt DK be to all the numbers that precede it, A, BG, C.

From DN take NL = BG, and NH = C.
a *hyp.* Becauſe DN. C (HN) *a* :: HN. BG (LN) *a* ::
b 17. 5. LN (BG.) A (KN.) *b* therefore by dividing
c 12. 5. each, ſhall DH. HN :: HL. LN :: LK. KN. *c*
d 3. *ax.* 1. wherefore DK C + BG + A :: LK (*d* BF.) KN (A.) *Which was to be dem.*

Coroll.

e 18. 5. Hence *e* by compounding. DN + BG + C. A + BG + C :: BG. A.

PROP. XXXVI.

1. A,2. B,4. C,8. D,16.
E,31. G,62. H,124. L,248. F,496.
M,31. N,465.
P---- Q----

If from a unit be taken how many numbers ſoever 1, A, B, C, D, in double proportion continually, until the whole added together E be a prime number; and if this whole

whole E multiplying the laſt produce a number F, that which is produced F ſhall be a perfect number.

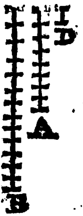
Take as many numbers E, G, H, L, likewiſe
in double proportion continually; then *a* of e- a 14. 7.
quality A. D :: E. L. *b* therefore $AL = DE$ *c* $=$ b 19. 7.
F. *d* whence $L = \frac{F}{2}$. Wherefore E, G, H, L, F, c *hyp.* d 7. *ax.* 7.
are ∺ in double proportion. Let $G - E$ be $= M$,
and $F - E = N$; *e* then M. E :: N. $E + G +$ e 35. 9.
$H + L$, *f* But $M = E$. *g* therefore $N = E +$ f 3. *ax.*
$G + H + L$. *b* therefore $F = 1 + B + C + D$ g 14. 5.
$+ E + G + H + L = E + N$. Moreover be- h 2. *ax.* 1.
cauſe D *k* meaſures DE (F) *l* therefore every k 7. *ax.* 7.
one, 1, A, B, C, *m* meaſuring D, as *m* alſo E, l 11. *ax.* 7.
G, H, L, does meaſure F. And further, no other m 11. 9.
number meaſures the ſaid F. For if there do, let n 9. *ax.* 7.
it be P, which meaſures F by Q. *n* therefore PQ o 19. 7.
$= F = D$. F. *o* therefore E. Q :: P. D. therefore p 13. 9.
ſeeing A a prime number meaſures D, *p* and ſo q 20. *def.* 7.
no other P meaſures the ſame, *q* conſequently
E does not meaſure Q. Wherefore E being ſup- r 31. 7.
poſed a prime number, *r* it ſhall be prime to Q. ſ 23. 7.
ſ wherefore E and Q are the leaſt in their pro- t 21. 7.
portion; *t* and ſo E meaſures P as many times u 13. 7.
as Q does D; *u* therefore Q is one of them A,
B, C. Let it be B, ſeeing then of equality B. x 19. 7.
D :: E. H :: *x* and ſo $BH = DE = F = PQ$. *x* y 14. 5.
and ſo alſo Q. B :: H. P. *y* therefore $H = P$.
therefore P is alſo one of them A, B, C, &c.
againſt the Hypotheſis. Wherefore no other be- z 22. *def.* 7.
ſide the foreſaid numbers meaſures F, and *z* con-
ſequently F is a perfect number. *Which was to be demonſtrated.*

The End of the ninth Book.

THE TENTH BOOK OF EUCLIDE'S ELEMENTS.

Definitions.

I. COmmenſurable magnitudes are thoſe, which are meaſured by one and the ſame meaſure.

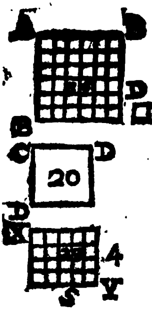


The note of commenſurability is ⊤⊏, *as A* ⊤⊏ *B; that is, the line A of* 8 *foot is commenſurable to the line B of* 13 *foot; becauſe D a line of one foot meaſures both A and B. Alſo* $\sqrt{18}$ ⊤⊏ $\sqrt{50}$; *becauſe* $\sqrt{2}$ *meaſures both* $\sqrt{18}$ *and* $\sqrt{50}$. *For* $\sqrt{\frac{18}{2}} = \sqrt{9} = 3$. *and* $\sqrt{\frac{50}{2}} = \sqrt{25} = 5$. *wherefore* $\sqrt{18}. \sqrt{50} :: 3. 5.$

II. Incommenſurable magnitudes are ſuch, of which no common meaſure can be found.

Incommenſurability is denoted by this mark ⊤⊏; *as* $\sqrt{6}$ ⊤⊏ $\sqrt{15}$ (5;) *that is,* $\sqrt{6}$ *is incommenſurable to the number* 5, *or to a magnitude deſigned by that number; becauſe there is no common meaſure of them, as ſhall appear hereafter.*

III. Right lines are commenſurable in power, when the ſame ſpace does meaſure their ſquares.



The mark of this commenſurability is ㄎ; as AB ㄎ CD. i. e. the line AB of 6 foot is in power commenſurable to the line CD, which is expreſſed by √ 20, becauſe E the ſpace of one foot ſquare does as well meaſure ABq (36) as the rectangle XY (20) to which the ſquare of the line CD (√ 20) is equal. The ſame note ㄎ ſometimes ſignifies commenſurable in power only.

IV. Lines incommenſurable in power are ſuch; to whoſe ſquares no ſpace can be found to be a common meaſure.

This incommenſurability is denoted thus; 5 ㄎ v √ 8 i. e. the numbers or lines 5, and v √ 8 are incommenſurable in power, becauſe their ſquares 25 and √ 8 are incommenſurable.

V. From which it is manifeſt, that to any right line given right lines infinite in multitude are both commenſurable and incommenſurable; ſome in length and power, others in power only. The right line given is called a Rational line.

The note of which is ῥ.

VI. And lines commenſurable to this line, whether in length and power, or in power only, are alſo called Rational, ῥ.

VII. But ſuch as are incommenſurable to it, are called Irrational.

And denoted thus ῤ.

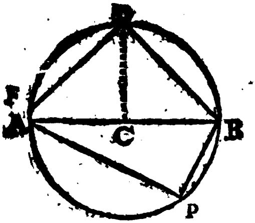
VIII. Alſo the ſquare which is made of the ſaid given right line is called Rational, ῥr.

IX. And likewiſe ſuch figures as are commenſurable to it, are Rational, ῥα.

X. But ſuch as are incommenſurable, Irrational, ῤα.

XI. And thoſe right lines alſo, which contain them in power, are Irrational ῤ.

Schol.



That the laſt ſeven definitions may be rendred more clear by an example, let there be a circle ADBP, whoſe ſemidiameter is CB, inſcribe therein the ſides of the ordinate figures, as of a Hexagone BP. of a triangle AP, of a ſquare BD, of a Pentagone FD. Therefore, if according to the 5. def. *the ſemidiameter CB be the Rational line given, expreſſed by the number* 2. *to which the other lines BP, AP, BD, FD, are to be compared, then BP a* $=$ *BC* $= 2$. *wherefore BP is* ῥ ⊔ *BC, according to the* 6. def. *Alſo AP b* $= \sqrt{12}$ *(for ABq* (16) − *BPq* (4) $=$ 12) *therefore AB* ῥ ⊔̍ *BC. likewiſe according to the* 6. def. *and APq* (12) *is* ῥν *by the* 9. def. *Moreover BD b* $= \sqrt{DCq + BCq} = \sqrt{8}$; *whence BD is* ῥ ⊔̍ *BC; and BDq* ῥν. *Laſtly, FDq* $= 10 - \sqrt{10}$. *(as ſhall appear by the praxis to be delivered at the* 10. 13.) *ſhall be* ῥν, *according to the* 10. def. *and conſequently FD* $= \sqrt{:10 - \sqrt{20}}$ *is* ῥ, *according to the* 11. def.

a cor. 15. 4.
b 47. 1.

A Poſtulate.

That any magnitude may be ſo often multiplied, till it exceed any magnitude whatſoever of the ſame kind.

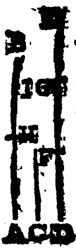
Axiomes.

1. A magnitude meaſuring how many magnitudes ſoever, does alſo meaſure that which is compoſed of them.

2. A magnitude meaſuring any magnitude whatſoever, does likewiſe meaſure every magnitude which that meaſures.

3. A magnitude meaſuring a whole magnitude and a part of it taken away, does alſo meaſure the reſidue.

PROP. I.

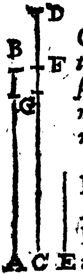


Two unequal magnitudes AB, C, being given, if from the greater AB there be taken away more than half (AH) and from the reſidue (HB) be again taken away more than half (HI) and this be done continually, there ſhall at length be left a certain magnitude IB, leſs than the leſs of the magnitudes firſt given C.

a Take C ſo often, till its multiple DE do ſomewhat exceed AB, and there be DE = FG = GE = C. Take from AB more than half HA, and from the remainder HB more than half HI, and ſo continually, till the parts AH, HI, IB, be equal in multitude to the parts DF, FG, GE. Now it is plain, that FE, which is not leſs than $\frac{1}{2}$ DE, is greater than HB, which is leſs than $\frac{1}{2}$ AB ⊐ DE. And in like manner GE, which is not leſs than $\frac{1}{2}$ FE, is greater than IB ⊐ $\frac{1}{2}$ HB. therefore C, or GE ⊐ IB. *Which was to be dem.* a *poſt.* 10.

The ſame may alſo be demonſtrated, if from AB the half AH be taken away, and again from the reſidue HB the half HI, and ſo forward.

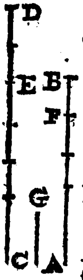
PROP. II.



Two unequal magnitudes being given (AB, CD) if the leſs AB, be continually taken from the greater CD, by an interchangeable ſubſtraction, and the reſidue do not meaſure the magnitude going before; then are the magnitudes given incommenſurable.

If it be poſſible, let ſome magnitude E be the common meaſure. Then becauſe AB taken from CD, as often as it can be, leaves a magnitude FD leſs than it ſelf, and FD taken from AB leaves GB, and ſo
a 1. 10. forward; *a* therefore at length ſome magnitude
b *hyp.* GB⊐E ſhall be left. therefore E *b* meaſuring AB,
c 2. *ax.*10. *c* and ſo CF, *b* and the whole CD, *d* ſhall alſo meaſure the reſidue FD. *c* conſequently alſo AG;
d 3. *ax.*10. *d* wherefore it ſhall likewiſe meaſure the remainder GB, leſs than it ſelf. *Which is abſurd.*

PROP. III.



Two commenſurable magnitudes being given AB, CD, to find out their greateſt common meaſure FB.

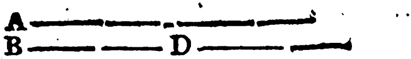
Take AB from CD, and the reſidue ED from AB, and FB from ED, till FB meaſure ED (which will come to paſs at length, *a* becauſe by the Hyp.
a 2. 10. AB ¬⊔ CD) FB ſhall be the magnitude required.

b *conſtr.* For FB *b* meaſures ED, *c* and ſo alſo
c 2. *ax.*10. AF; but it meaſures it ſelf too, *d* therefore likewiſe AB, *c* and conſequently
d 1. *ax.*10. CE, *d* and ſo the whole CD. Wherefore FB is the common meaſure of AB, CD, If you affirm G to be a common meaſure greater than that,
e 2. *ax.*10. then G meaſuring AB and CD, *e* meaſures alſo
f 3. *ax.*10. CE and *f* the remainder ED, *e* and ſo AF; and *f* conſequently the remainder FB, the greater the leſs. *Which is abſurd.*

Coroll.

Hence, A magnitude that meaſures two magnitudes, does alſo meaſure their greateſt common meaſure.

PROP. IV.



Three commenſurable magnitudes being given A, B, C, to find out their greateſt common meaſure.

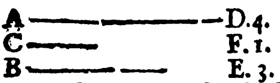
a Find out D the greateſt common meaſure of any two A, B; *a* alſo E the greateſt common meaſure of D and C. therefore E is the magnitude ſought for. a 3. 10.

a For it is clear, E meaſuring D and C, *b* does meaſure the three A, B, C. Conceive another magnitude F greater than that to meaſure them; *c* then F meaſures D, *c* and conſequently E the greateſt common meaſure of D, and C, the greater the leſs. *Which is abſurd.* b *conſtr.* & 2. *ax.* 10. c *cor.* 3. 10.

Coroll.

Hence alſo it appears, that if a magnitude meaſure three magnitudes, it ſhall likewiſe meaſure their greateſt common meaſure.

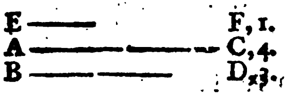
PROP. V.



Commenſurable magnitudes A, B, have ſuch proportion one to another, as number hath to number.

a C being found the greateſt common meaſure of A, B; as often as C is contained in A and B, ſo often is 1 contained in the numbers D and E; *b* therefore C.A :: 1.D; wherefore inverſely A.C :: D. 1. *b* but likewiſe C. B :: 1.E. *c* therefore of equality A. B :: D.E :: N. N. *W. W. to be dem.* a 3. 10. b 17. *def.* 7.

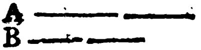
PROP. VI.



If two magnitudes A, B, have ſuch proportion one to another as number C has to number D, thoſe magnitudes A, B ſhall be commenſurable.

a *ſch.*10.6. What part 1 is of the number C, *a* that let E
b *conſtr*, be of A. Therefore becauſe E.A *b* :: 1.C. and A.B
c *hyp*, *c* :: C.D. *d* therefore of equality ſhall E.B :: 1.D.
d 21. 5. Wherefore ſeeing 1 *e* meaſures the number D, *f*
e 5.*ax*.7. likewiſe E meaſures B; but it *g* alſo meaſures A,
f 20.*def*.7. *h* therefore A ⌝⊔ B. *Which was to be dem.*
g *conſtr.*
h 1.*def*.10.

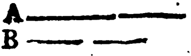
PROP. VII.



Incommenſurable magnitudes A, B, have not that proportion one to another, which number has to number.

a 6, 10. If you affirm A. B :: N. N. *a* then A ⌝⊔ B *againſt the Hypotheſis.*

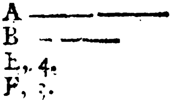
PROP. VIII.



If two magnitudes A, B, have not that proportion one to another, which number has to number, thoſe magnitudes are incommenſurable.

a 5. 10. Conceive A ⌝⊔ B *a* then A. B :: N. N. *contrary to the Hypotheſis.*

PROP. IX.



The ſquares deſcribed of right lines commenſurable in length, have that proportion one to another, that a ſquare number has to a ſquare number. And ſquares, which have that proportion

portion one to another, that a ſquare number has to a ſquare number, ſhall alſo have their ſides commenſurable in length. But ſuch ſquares as are made of right lines incommenſurable in length, have not that proportion one to another, which a ſquare number has to a ſquare number. And ſquares which have not ſuch proportion one to another as a ſquare number has to a ſquare number, have not their ſides commenſurable in length.

1. *Hyp.* A. ⌐□ B. I ſay Aq. Bq :: Q. Q.

For *a* let A.B :: number E. number F. therefore a 5. 10.
$\frac{Aq}{Bq}$ (*b* $\frac{A}{B}$ twice) *c* = $\frac{E}{F}$ twice. *d* = $\frac{Eq}{Fq}$. *e* there- b 20. 6.
fore Aq. Bq :: Eq. Fq :: Q. Q. *Which was to be dem.* c *ſch.*23.5.

2. *Hyp.* Aq. Bq :: Eq. Fq :: Q. Q. I ſay A ⌐□ B. For d 11. 8.
$\frac{A}{B}$ twice (*f* $\frac{Aq}{Bq}$) *g* = $\frac{Eq}{Fq}$ *h* = $\frac{E}{F}$ twice. *i* there- e 11. 5.
fore A. B :: E. F :: N. N. *k* wherefore A ⌐□ B. f 20. 6.
Which was to be dem. g *hyp.*
h 11. 8.
i *ſch.*23.5.
k 6. 10.

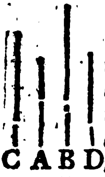
3. *Hyp.* A ⌐□ C. I deny that Aq. Bq :: Q. Q. For ſuppoſe Aq. Bq :: Q. Q. then A ⌐□ B, as is ſhewn before, *againſt the Hyp.*

4. *Hyp.* Not Aq. Bq :: Q. Q. I ſay that A ⌐□ B, For conceive A ⌐□ B. then Aq. Bq :: Q. Q. as above, *againſt the Hyp.*

Coroll.

Lines ⌐□ are alſo ⌐⊐. but not on the contrary. And lines ⌐□ are not therefore ⌐⊐. but ⌐⊐ are alſo ⌐□.

PROP. X.



If four magnitudes be proportional (C. A :: B. D) and the firſt C be commenſurable to the ſecond A, the third B ſhall be commenſurable to the fourth D. And if the firſt C be incommenſurable to the ſecond A, alſo the third B ſhall be incommenſurable to the fourth D.

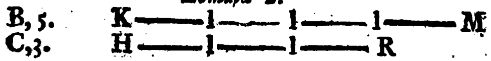
a 5. 10. If C ⊔ A, *a* then C. A :: N. N *b* :: B. D: *b*
b 6. 10. therefore B ⊔ D. But if ⊔ A, *c* then ſhall
c 7. 10. not C. A :: N. N :: B. D. *d* wherefore B ⊔ D:
d 8. 10. *Which was to be dem.*

Lemma 1.

To find out two plane numbers, not having the proportion which a ſquare number hath to a ſquare.

Any two plane numbers not like will ſatisfy this Lemma, as thoſe numbers which have ſuperparticular, ſuperbipartient, or double proportion; or any two prime numbers. *See ſch.* 27. 8.

Lemma 2.



To find out a line HR, to which a right line given KM hath the proportion of two numbers given B, C.

a *ſch.* 10. 6. *a* Divide KM into as many equal parts as there are units in the number B. and let as
b 3. 1. many of theſe, as there are units in the number C, *b* make the right line HR, it is manifeſt that KM. HR :: B. C.

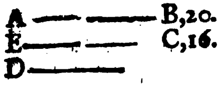
Lemma 3.

To find out a line D, to the ſquare of which, the
a 2. *lem.* 10. *ſquare of a right line given KM hath the proportion*
10. *of two numbers given B, C.*

b 13. 6. Allow B. C *a* :: KM. HR. and between KM
c 20. 6. and HR, *b* find a mean proportional D. There-
d *conſtr.* fore KMq. Dq *c* :: KM. HR *d* :: B. C.

PROP.

PROP. XI.

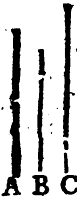


To find two right lines incommenſurable to a right line given A, one D in length only, the other E in power alſo.

1. Take the numbers B, C, *a* ſo that there be not B.C :: Q.Q. *b* and let B, C :: Aq. Dq. *c* it is plain that A ⊓ D. But Aq *d* ⊓ Dq. *Which was to be done.*
a 1.*lem.*10.
10.
b 3.*lem.*10.
10.

2. *d* Make A.E :: E.D. I ſay Aq ⊓ Eq. For A. D :: Aq.Eq. therefore ſince A ⊓ D, as before : *f* therefore Aq ⊓ Eq. *Which was to be done.*
c 9. 10.
d 6. 10.
d 13. 6.
e 20. 6.
f 10. 10.

PROP. XII.



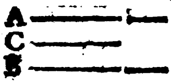
Magnitudes (A, B) commenſurable to the ſame magnitude C, are alſo commenſurable one to the other.

Becauſe A ⊓ C, and C ⊓ B, *a* let A. D,18. E,8. C :: N.N :: D. E, and C. F, 2. G,3. B :: N. N :: F. G. *b* take H,5.I,4.K,6. three numbers H. I, K, the leaſt ÷ in the proportions of D to E, and F to G. Now becauſe A.C *c* :: D.E *c* :: H.I, and C. B *c* :: F.G :: I.K. *d* therefore of equality A.B :: H.K :: N. N, *e* therefore A ⊓ B. *Which was to be dem.*
a 5. 10.
b 4. 8.
c *conſtr.*
d 22. 5.
e 6. 10.

Schol.

Hence, Every right line commenſurable to a rational line is alſo it ſelf rational. And all right lines rational are commenſurable to one another, at leaſt in power. Alſo, every ſpace commenſurable to a rational ſpace is rational too : and all rational ſpaces are commenſurable one to another. But magnitudes whereof one is rational, the other irrational, are incommenſurable amongſt themſelves.
12.10. *and def.* 6.
def. 9.
def. 7. & 10.

PROP. XIII.



If there be two magnitudes A, B, and one of them A commenſurable to a third C, but the other B incommenſurable, thoſe magnitudes A, B are incommenſurable.

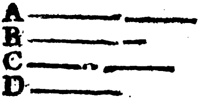
a *hyp.* Conceive B ⊤⊔ A. then being C *a* ⊤⊔ A, *b*
b 12. 10. therefore C ⊤⊔ B, *againſt the Hyp.*

PROP. XIV.

If there be two magnitudes commenſurable A,B; and one of them A incommenſurable to any other magnitude C, the other alſo B ſhall be incommenſurable to the ſame C.

a *hyp.* Imagine B ⊤⊔ C. then for that A *a* ⊤⊔
b 12. 10. B, *b* therefore A ⊤⊔ C, *againſt the Hyp.*

PROP. XV.

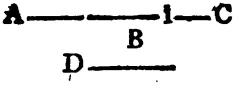


If four right lines be proportional (A. B :: C. D.) and the firſt A be in power more than the ſecond B by the ſquare of a right line commenſurable to it ſelf in length, then alſo the third C ſhall be more in power than the fourth D by the ſquare of a right line commenſurable to it ſelf in length. But if the firſt A be more in power than the ſecond B by the ſquare of a right line incommenſurable to it ſelf in length, then ſhall the third C be more in power than the fourth D by the ſquare of a right line incommenſurable to it ſelf in length.

a *hyp.* For becauſe A.B *a* :: C. D. *b* therefore Aq. Bq ::
b 22. 6. Cq.Dq. *c* therefore by diviſion Aq – Bq. Bq :: Cq
c 17. 5. – Dq.Dq. *d* wherefore √: Aq—Bq. B :: √: Cq—
d 22. 6. Dq.D. *e* and ſo inverſely B.√: Aq – Bq :: D.√:Cq
e *cor.* 4. 5. – Dq. *f* therefore of equality A.√: Aq – Bq :: C.
f 22. 5. √:Cq – Dq. conſequently if A ⊤⊔, or ⊤⊔ √ Aq – Bq,

—Bq. *g* then likewiſe C ⊓, or ⊓ √ : Cq — *g* 10. 10.
Dq. *Which was to be dem.*

PROP. XVI.



If two magnitudes commenſurable AB, BC, be compoſed, the whole magnitude AC ſhall be commenſurable to each of the parts AB, BC. And if the whole magnitude AC be commenſurable to either of the parts AB, or BC, thoſe two magnitudes given at firſt AB, BC, ſhall be commenſurable.

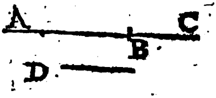
1. *Hyp.* *a* Let D be the common meaſure of AB, BC; *b* ſo D meaſures AC. and therefore AC ⊓ AB, and BC. *Which was to be dem.* a 3. 10. b 1. ax. 10. c 1. def. 10.

2. *Hyp.* *a* Let D be the common meaſure of AC, AB. *d* therefore D meaſures AC — AB (BC) and conſequently AB ⊓ BC. *Which was to be dem.* d 3. ax. 10.

Coroll.

Hence it follows, if a whole magnitude compoſed of two, be commenſurable to any one of them, the ſame ſhall be commenſurable to the other alſo.

PROP. XVII.



If two incommenſurable magnitudes AB, BC, be compoſed, the whole magnitude alſo AC ſhall be incommenſurable to either of the two parts AB, BC. And if the whole magnitude AC be incommenſurable to one of them AB, the magnitudes firſt given AB, BC, ſhall be incommenſurable.

1. *Hyp.* If it can be, let D be the common meaſure of AC, AB. *a* therefore D meaſures AC — AB (BC) *b* and therefore alſo AB ⊓ BC, *againſt the Hypoth.* a 3. ax. 10. b 1. def. 10.

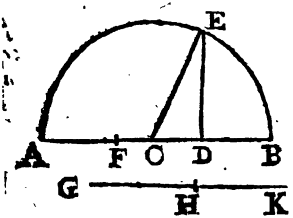
2. *Hyp.*

c 16. 10. 2. *Hyp.* Conceive AB ⊤⊏ BC. *c* therefore AC ⊤⊏ AB, *againſt the Hyp.*

Coroll.

Hence alſo, If one magnitude, compoſed of two, be incommenſurable to any one of them, the ſame alſo ſhall be incommenſurable to the other.

PROP. XVIII.



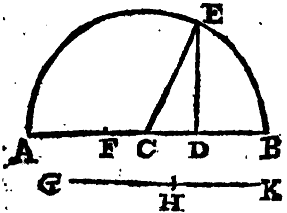
If there be two unequal right lines AB, GK and upon the greater AB a parallelogram ADB equal to the fourth part of a ſquare made of the leſs line GK, and wanting in figure by a ſquare, be applied, and divide the ſaid AB into parts commenſurable in length AD, DB ; then ſhall the greater line AB be more in power than the leſs GK by the ſquare of a right line FD commenſurable in length to the greater. And if the greater AB be in power more than the leſs GK by the ſquare of the right line FD commenſurable unto it ſelf in length, and a parallelogram ADB equal to the fourth part of the ſquare made of the leſs line GK, and wanting in figure by a ſquare, be applied to the greater AB, then ſhall it divide the ſame into parts AD, DB commenſurable in length.

a 10. 1. b 28. 6. c 8. 2. d conſtr. & 4. 2. e 16. 10. g con. 16. 10. k 12. 10. l 16. 10.

a Divide GK equally in H, and *b* make the rectangle ADB   GHq. Cut off AF — DB. then is ABq *c* = 4 ADB *d* (4 GHq or GKq) + FD. Now in the firſt place, if AD ⊤⊏ DB, then ſhall AB *e* ⊤⊏ BD *c* ⊤⊏ 2 DB *f* (AF   DB, or AB—FD) *b* therefore AB ⊤⊏ FD. *Which was to be dem.* But ſecondly, if AB ⊤⊏ FD, *b* then ſhall AB ⊤ AB — FD (2 DB) *k* therefore AB   DB. *l* wherefore AD ⊤⊏ DB. *Which was to be dem.*

PROP.

PROP. XIX.

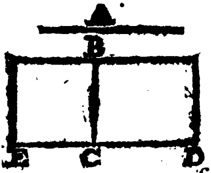


If there be two right lines unequal AB, GK, and to the greater AB be applied a parallelogram ADB equal to the fourth part of a ſquare made upon the leſs GK, and wanting in figure by a ſquare, and alſo thus applied divide the ſaid AB into parts AD, DB incommenſurable in length; the greater line AB ſhall be in power more than the leſs GK by the ſquare of the right line FD incommenſurable to the greater in length. And if the greater line AB be more in power than the leſs GK by the ſquare of a right line FD incommenſurable unto it ſelf in length, and if alſo upon the greater AB be applied a parallelogram ADB equal to the fourth part of the ſquare of the leſs GK and wanting in figure by a ſquare, then ſhall it divide the ſaid greater line AB into parts incommenſurable in length AD, DB.

Suppoſe all the ſame that was done and ſaid
in the prec. Prop. Therefore firſt, If AD ˥⊔
DB, *a* then ſhall AB ˥⊔ DB. *b* Wherefore AB a 17. 10.
˥⊔ 2 DB (AB — FD) *c* therefore AB ˥⊔ FD. b 13. 10.
Which was to be dem. c *cor.* 17. 10.

Secondly, If AB ˥⊔ FD, then AB ˥⊔ AB—
FD (2 DB;) *d* wherefore AB ˥⊔ DB, *e* and con- d 13. 10.
ſequently AD ˥⊔ DB. *Which was to be dem.* e 17. 10.

PROP. XX.

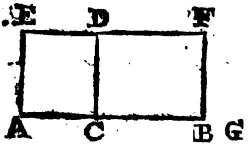


A rectangle BD comprehended under right lines BC, CD, rational and commenſurable in length, according to one of the foreſaid ways, is rational.

Let A be given ρ̇, and
a 46. 1. a the ſquare BE deſcribed upon BC. Becauſe DC.
b 1. 6. CE (BC) *b* :: BD. BE. and DC *c* ⊓⊔ BC, *d*
c *hyp.* therefore ſhall the rectangle BD be ⊓⊔ ſquare
d 10. 10. BE. wherefore ſeeing the ſquare BE *e* ⊓⊔ Aq,
e *hyp. and* ſhall alſo BD be ⊓⊔ Aq. and ſo the rectangle
9 *def.* 10. BD is ρ̇ν. *Which was to be dem.*
f 12. 12. Note, *There are three kinds of lines rational commenſurable one to another. For either, of two lines rational commenſurable in length one to the other, one is equal to the rational line propounded, or neither of them is equal to it, notwithſtanding both of them are commenſurable to it in length; or laſtly, both of them are commenſurable to the rational line given only in power. And theſe are the ways which the preſent Theorem ſpeaks of.*

In numbers, let there be BC $\sqrt{8}$ ($2\sqrt{2}$) and CD $\sqrt{18}$ ($3\sqrt{2}$) then ſhall the rectangle BD $= \sqrt{144} = 12$.

PROP. XXI.



If a rational rectangle DB be applied to a rational line DC, it makes the breadth thereof CB rational, and commenſurable in length to that
a 1. 6.
b *hyp.* *line DC, whereto DB is applied.*
c *ſch.* 12. Let G be propounded ρ̇, and the ſquare DA
10. deſcribed on BC. becauſe BD. DA :: *a* BC. CA;
d 10. 10. and BD, DA *b* are ρ̇α *c* and ſo ⊓⊔. *d* therefore
e *ſch.* 12. BC ⊓⊔ CA. but CD (CA) is ρ̇. *e* therefore BC
10. is ρ̇. *Which was to be dem.*

In numbers, let there be rectangle DB. 12, and DC, √ 8. then ſhall CB, √ 18. but √ 18 = 3 √ 2. and √ 8 = 2 × √ 2.

Lemma.

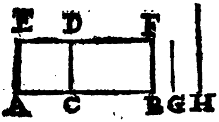
A ———
B ————
C ———

To find out two right lines rational commenſurable only in power.

Let A be propounded $\dot{\rho}$. *a* Take B ⊤̲ A, *a* and C ⊤̲ B. *b* it is clear that B and C are the lines required.

a 11. 10.
b *ſch.* 12. 10.

PROP. XXII.



A rectangle DB comprehended under right lines rational DC, CB, commenſurable in power only, is irrational: and the right line H, which containeth that rectangle in power is irrational, and called a Medial line.

Let G be the propounded $\dot{\rho}$, and the ſquare DA deſcribed on DC, and let Hq=DB. Becauſe AC. CB *a* :: DA. DB. *b* and AC ⊓̲ CB, *c* ſhall be DA ⊓̲ DB (Hq,) *d* but Gq ⊓̲ DA. *e* therefore Hq ⊓̲ Gq. *f* wherefore H is $\dot{\rho}$. *Which was to be dem.* and let it be called a Medial line, becauſe AC. H :: H. CB.

a 1. 6.
b *hyp.*
c 10. 10.
d *hyp.* and 9 3 *def.* 10.

In numbers, let there be DC, 3. and CB, √ 6. then ſhall the rectangle be DB (Hq) √ 54. wherefore H is *v* √ 54.

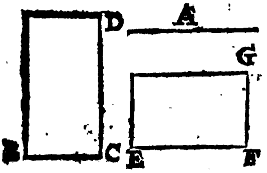
e 13. 10.
f 11. 10.

The note of a medial line is *μ*, of a medial rectangle *μι*, of more together *μα*.

Schol.

Every rectangle that can be contained under two right lines rational commenſurable only in power, is medial, altho' it be contained under two right lines irrational: and every medial rectangle may be contained under two right lines rational, commenſurable only in power, as for example, the $\sqrt{24}$ is $\mu\nu$, becauſe it is contained under $\sqrt{3}$, and $\sqrt{8}$, which are $\dot{\rho}$, ⊤⊥. altho' it may be contained under $v\sqrt{6}$, and $v\sqrt{96}$ irrationals; for $\sqrt{24} = v\sqrt{576} = v\sqrt{6} \times v\sqrt{96}$.

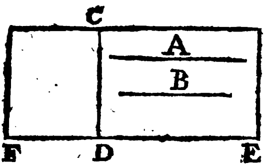
PROP. XXIII.



If the rectangle BD made of a medial line A, be applied on a rational line BC, it makes the breadth CD rational, and incommenſurable in length to the line BC, whereunto the rectangle BD is applied.

a ſch. 12. Becauſe A is μ, *a* therefore ſhall Aq be equal
10. to ſome rectangle (EG) contained under EF and
b 1. ax. 1. FG $\dot{\rho}$ ⊤⊥. *b* therefore BD = EG. *c* whence BC.
c 14. 6. EF :: FG. CD. *d* therefore BCq.EFq :: FGq.CDq.
d 22. 6. But BCq and EFq *e* are $\dot{\rho}\alpha$, *f* and ſo ⊤⊔. *g* there-
e hyp. fore FGq ⊤⊔ CDq. Wherefore being FG is $\dot{\rho}$,
f ſch. 12. *h* therefore CD ſhall be $\dot{\rho}$. Moreover, becauſe
10. EF.FG *k* :: EFq.EG (BD;) for that EF ⊤⊔ FG,
g 10. 10. *l* ſhall EFq be ⊤⊔ BD. But EFq *m* ⊤⊔ CDq. *n*
h ſch. 12. therefore the rectangle BD ⊤⊔ CDq. Whence
10. being CDq. BD *o* :: CD. BC. *p* ſhall CD be ⊤⊔
k 1. 6. BC. therefore, &c.
l 10. 10.
m ſch. 12.
10.
n 13. 10.
o 1. 6.
p 10. 10.

PROP. XXIV.



A right line B commenſurable to a medial line A is alſo a medial line.

Upon CD $\dot{\rho}$ *a* make the rectangle
a 11. 6. CE = Aq; *a* and the
b hyp. rectangle CF = Bq. Becauſe Aq (CE) is $\mu\nu$, *b* and
c 23. 10. CD $\dot{\rho}$. *c* therefore ſhall the latitude DE be $\dot{\rho}$ ⊤⊥
d 1. 6. CD. But for that CE.CF *d* :: ED.DF. and CE *e* ⊤⊔
e hyp. CF,

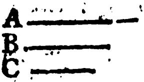
CF, *f* therefore ED ⊓ DF. *g* therefore DF is f 10. 10.
ρ ⊓ CD. *h* whence the rectangle CF (Bq) is g 12. *and*
μ, and ſo B is μ. *Which was to be dem.* 13. 10.

Obſ. *That the note* ⊔ *for the moſt part ſignifies* h 22. 10.
commenſurable in power only, as in this and the precedent demonſtrations, &c.

Coroll.

Hereby it is manifeſt that a ſpace commenſurable to a medial ſpace, is alſo medial.

Lemma.

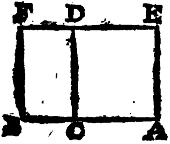


To find out two right lines medial A, B, commenſurable in length, and alſo two, A, C, commenſurable only in power.

a Let A be any μ, *b* take B ⊓ A, and *c* C a *lem.* 22.
⊔ A. *d* it appears to be done. 10. *and*
13. 6.

PROP. XXV.

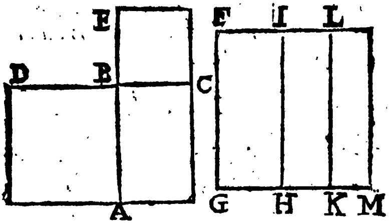
b 2 *lem.* 10.
10.



A rectangle DB contained under DC, CB medial right lines commenſurable in length, is medial. c 3 *lem.* 10. 10. d *conſtr.* & 24. 10.

Upon DC deſcribe the ſquare DA. Being AC (DC.) CB *a* ∷ a 1. 6.
DA. DB. and DC ⊓ CB; *b* ſhall DA ⊓ DB. b 10. 10.
c therefore DB is μ. *Which was to be dem.* c 24. 10.

PROP. XXVI.



A rectangle AC comprehended under medial right lines AB, BC commensurable only in power, is either rational or medial.

a 46. 1. Upon the lines AB, BC, *a* deſcribe the ſquares AD, CE; and upon FG ρ̇ make the rectangles FH,
b *cor.*16.6. = AD, *b* and IK=AC. *b* and LM=CE.

The ſquares AD, CE, that is, the rectangles
c *hyp.* & FH, LM, *c* are μα and ⊤⊔. therefore GH, KM,
24. 10. having the ſame proportion *d* are ρ̇, *e* and ⊤⊔.
d 23. 10. *f* therefore GH × KM is ρ̇ν. But becauſe AD,
e 10. 10. AC, CE. that is, FH, IK, LM, *g* are ∺; *h* and
f 20. 10. ſo GH, HK. KM alſo ∺; *k* thence HKq=GH
g *ſch.*22.6. × KM. *l* therefore HK is ρ̇ or ⊤⊔, or ⊤⊔̣ IH
h 1. 6. (GF;) if ⊤⊔, *m* then the rectangle IK or AC is
k 17. 6. ρ̇ν. but if ⊤⊔̣, *n* then AC is μν. *Which was to*
l 12. 10. *be dem.*
m 20. 10.
n 22. 10.

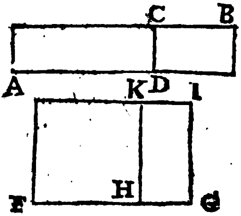
Lemma.

If A and E be ⊤⊔̣ only, Then first ſhall Aq,
a *hyp. and Eq, Aq+Eq, Aq−Eq a* ⊤⊔. *And ſecondly* Aq, Eq,
10. 10. Aq+Eq. Aq−Eq ⊤⊔ AE and 2 E. For A.E *b* ∷
b 1. 6. Aq. AE *b* ∷ AE. Eq. therefore ſeeing A *c* ⊤⊔ E,
c *hyp.* *d* ſhall Aq ⊤⊔ AE, *e* and 2 AE. alſo Eq *d* ⊤⊔
d 10. 10. AE, *e* and 2 AE. wherefore becauſe Aq+Eq ⊤⊔
e 14. 10. Aq and Eq; and Aq − Eq ⊤⊔ Aq and Eq. *f*
f 14. 10. therefore ſhall Aq + Eq. *f* and Aq − Eq be ⊤⊔ AE, and 2 AE.

Hence alſo thirdly, Aq. Eq. Aq + Eq, Aq − Eq, 2 AE *g* ⊔ Aq + Eq † 2 AE; and Aq + Eq − 2 AE *g* and Aq † Eq + 2 AE ⊔ Aq + Eq − 2 AE. *h* (Q. A − E.)

g 14. 16.
& 17. 10.
h cor. 7. 12.

PROP. XXVII.

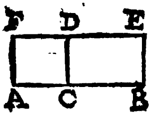


A medial rectangle AB exceedeth not a medial rectangle AC by a rational rectangle DB.

Upon EF ρ̇, *a* make EG = AB, *a* and EH = AC. The rectangles AB, AC, *i. e.* EG, EH, *b* b *hyp.* are μα; *c* therefore FG and FH are ρ̇ ⊔̸ EF. Whence, if KG, *d* *i. e.* DB be ρ̇v. *e* then ſhall HG be ⊔ HK; *f* wherefore HG ⊔̸ FH. *g* and conſequently FGq ⊔ FHq. But FH is ρ̇. *h* therefore is FG ρ̇. but FG was ρ̇ before. *Which is contradictory.*

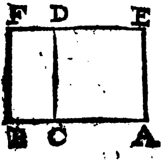
a cor. 16. 6.
c 23. 10.
d 3. ax. 1.
e 21. 10.
f 13. 10.
g *lem.* 16. 10.
b *ſch.* 12. 10.

Schol.



1. *A rational rectangle* AE *exceeds a rational rectangle* AD *by a rational rectangle* CE.

For AE *a* ⊔ AD. *b* therefore AE ⊔ CE. *c* wherefore CE is ρ̇v. *Which, &c.*

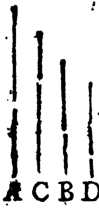


2. *A rational rectangle* AD *joyned with a rational rectangle* CF *makes a rational rectangle AF.*

For AD *a* ⊔ AF. *b* wherefore AF ⊔ AD and CF, *c* and ſo AF is ρ̇v. *Which was to be dem.*

a *hyp.*
b *cor.* 16. 10.
c *ſch.* 12. 10.
a *ſch.* 12. 10.
b 16. 10.
c *ſch.* 12. 10.

PROP. XXVIII.

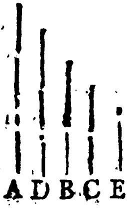


To find out medial lines (C and D,) which contain a rational rectangle CD.

a Take A and B ρ̇ ⊤⊥, *b* make A.C::
a *lem.* 21. C.B. *c* and A.B :: C.D. I ſay the thing
10. required is done. For AB (Cq) *d* is μν,
b 13. 6. *d* whence C is μ. but being that A. B *e*
c 12. 6. :: C,D. *f* therefore C ⊤⊥ D. *g* and con-
d 12. 10. ſequently D is μ. Moreover by inver-
e *conſtr.* ſion A.C :: B.D. *i.e.* C.B :: B.D. *h* there-
f 10. 10. fore Bq = CD. But Bq is ρ̇ν. *h* therefore CD is ρ̇ν.
g 14. 10. *Which was to be done.*

h 17. 6. In numbers, let A be $\sqrt{2}$; and B $\sqrt{6}$. there-
k *ſch.* 12. fore C is $v\sqrt{12}$. make $\sqrt{2}$. $\sqrt{6}$ :: $v\sqrt{12}$. D,
10. or $v\sqrt{4}$. $v\sqrt{36}$ :: $v\sqrt{12}$. D. then ſhall D be $v\sqrt{108}$. but $v\sqrt{12} \times v\sqrt{108} = v\sqrt{1296} = \sqrt{36}$, $= 6$. therefore CD is 6. likewiſe C. D :: 1. $\sqrt{3}$. wherefore C ⊤⊥ D.

PROP. XXIX.



To find out medial right lines commenſurable in power only, D and E, containing a medial rectangle DE.

a Take A, B, C, ρ̇ ⊤⊥. make A.D
a *lem.* 21. *b* :: D. B. *c* and B. C :: D. E. I ſay
10. the thing deſired is performed.
b 13. 6. For AB *d* = Dq. and AB *e* is μν,
c 12. 6. therefore D is μ; and B *f* ⊤⊥ C,
d 17. 6. *g* whence D ⊤⊥ E. therefore *h* E is
e 22. 10. μ. Moreover B, C *f* :: D. E. and by inverſion B.
f *conſtr.* & D :: C. E. *i.e.* D. A :: C. E. *l* therefore DE =
g 10. 10. AC. But AC *m* is μν. therefore DE is μν. *Which*
h 24. 10. *was to be done.*
k *conſtr.* &
cor. 4. 5.

In numbers, let A be 20. and B, $v\sqrt{200}$, and C,
l 16. 6. $\sqrt{80}$. Therefore D is $\sqrt{\sqrt{80000}}$; and E $v\sqrt{12800}$. Therefore DE $= \sqrt{\sqrt{1024000000}} =$
m 22. 6. 32000. and D.E :: $\sqrt{10}$. 2. wherefore D ⊤⊥ E.

Schol.

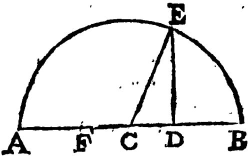
A,6. C,12.
B,4. D,8.
AB,24. CD,96.

A,6. C,5.
B,4. D,8.
AB,24. CD,40.

To find out plane numbers, like or unlike.

Take any four numbers proportional A. B :: C. D. it is manifeſt that AB and CD are like plane numbers. And you may find out as many unlike plane numbers, as you pleaſe, by help of *Schol.* 27. 8.

Lemma.



To find out two ſquare numbers (DEq and CDq) ſo that the number compoſed of them (CEq) be ſquare alſo.

Take AD, DB like plane numbers (of which let both be equal, or both odd) *viz.* AD, 24. and DB, 6. The total of theſe (AB) is 30; the difference (FD) 18 half of which (CD) is 9. *a* Now the like plane numbers AD, DB, have a 18. 8.
one mean number proportional, namely DE. therefore it is evident that every of thoſe numbers CE, CD, DE, are rational, and by conſequence CEq (*b* CDq + DEq) is the ſquare b 47. 1.
number required.

Whereby it will be eaſy to find out two ſquare numbers, the exceſs of which is a ſquare or not a ſquare number, namely by the ſame conſtruction *c* ſhall CEq — CDq be = DEq. c 3, *ax.* 1.

But if AD, DB be plane numbers unlike, the medial proportional line (DE) ſhall not be a rational number, and ſo neither ſhall the exceſs (DEq) of the ſquare numbers, CEq, CDq, be a ſquare number.

Lemma 2.

2. To find out two ſuch ſquare numbers B, C, as the number compounded of them D is not ſquare. Alſo to divide a ſquare number A into two numbers B, C, not ſquares.

A, 3. B, 9. C, 36. D, 45.

1. Take any ſquare number B, and let C be = 4 B, and D = B + C. I ſay the matter is done.

For B is Q. by the conſtr. likewiſe becauſe
a 24. 8. B.C :: 1.4 :: Q. Q. *a* therefore C alſo ſhall be a
ſquare number. But becauſe B + C. (D) C ::
b cor. 24. 8. 5. 4 :: not Q, Q. *b* therefore ſhall not D be a
ſquare number. *Which was to be done.*

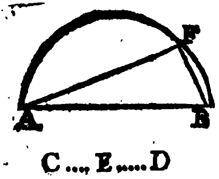
A, 36. B, 24. C, 12. D, 3. E, 2. F, 1.

2. Let A be ſome ſquare number. Take D, E, F, plane numbers alike, and let D be = E + F. make D. E :: A. B. and D. F :: A. C. I ſay the thing required is done.

For becauſe D. E + F :: A. B + C. and D =
a 14. 5. E + F. *a* therefore ſhall A = B + C. Now ſup-
b 21. *def.* 7. poſe B to be ſquare, *b* then A and B, *c* and con-
c 26. 8. ſequently D and E are like plane numbers.
Which is contrary to the Hyp.

The ſame abſurdity will follow if C be ſuppoſed a ſquare number. Therefore, &c.

PROP. XXX.



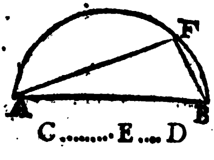
To find out two ſuch rational right lines AB, AF, commenſurable only in power, as the greater AB ſhall be in power more than the leſs AF by the ſquare of a right line BF commenſurable in length to the greater.

Let AB be the line given ῥ. *a* Take the ſquare a 1. *lem.*
numbers CD, CE, ſo that CD−CE (ED) be not 29. 10.
Q. *b* and let there be CD. ED :: ABq. AFq. In b 3. *lem.*
a circle deſcribed upon the diameter AB *c* draw 10. 10.
AF, and alſo BF. Then I ſay AB, AF, are the c 1. 4.
lines required. d *conſtr.*

For ABq.AFq *d* :: CD.ED, *e* therefore ABq ⊤⊥ e 6. 10.
AFq. but AB is ῥ. *f* therefore AF is alſo ῥ. But f *ſch.* 12.
becauſe CD is Q: and ED not Q: *g* therefore ſhall 10.
AB be ⊤⊥ AF. Moreover by reaſon of the *h* right g 9. 10.
angle AFB, is ABq *k* = AFq + BFq; therefore h 3. 1. 3.
ſeeing ABq.AFq :: CD.ED. by converſion of pro- k 47. 1.
portion ſhall ABq.BFq :: CD.CE :: Q.Q. *l* there- l 9. 10.
fore AB ⊤⊥ BF. *Which was to be done.*

In numbers, let there be AB,6; CD,9; CE,4; wherefore ED,5. Make 9. 5 :: 36. (Q:6) AFq. then AFq ſhall be 20. and conſequently AF √20. therefore BFq = 36 − 20 = 16. wherefore BF is 4.

PROP. XXXI.



To find out two rational lines AB, AF commenſurable only in power, ſo that the greater AB ſhall be in power more than the leſs AF by the ſquare of a right line BF incommenſurable in length to the greater.

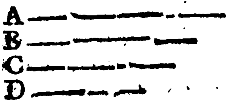
Let AB be the line given ῥ. *a* Take the ſquare a 2. *lem.*
numbers CE, ED, ſo that CD = CE + ED be 19. 10.
not Q. and in the reſt follow the conſtruction of
the preced. Prop. I ſay then the thing required
is done.

For

b 9. 10. For, as above, AB, AF, are ρ̇ ⊤⊔. alſo ABq. BFq :: CD. ED. therefore being CD is not Q. AB, BF *b* ſhall be ⊓. *Which was to be done.*

In numbers, let there be AB, 5. CD, 45. CE = 36. ED = 9. Make 45. 9 :: 25 (ABq.) 5 (AFq.) therefore AF = √ 5. conſequently BFq = 45 − 25 = 20. wherefore BF = √ 20.

PROP. XXXII.

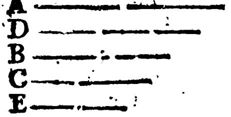


To find out two medial lines C, D, commenſurable only in power, comprehending a rational rectangle CD, ſo that the greater C be more in power than the leſſer D by the ſquare of a right line commenſurable in length to the greater.

a 30. 10. *a* Take A and B ρ̇ ⊤⊔; ſo as √ Aq − Bq ⊓
b 13. 6. A. *b* and make A. C :: C. B, *c* and A. B :: C.D.
c 12. 6. I ſay the thing is done.
d *conſtr.* For becauſe A and *d* B are ρ̇ ⊤⊔. *e* therefore
e 22. 10. ſhall C (*f* √ AB) be *μ*. *g* and thence alſo C ⊤⊔ D.
f 17. 6. *h* therefore D is likewiſe *μ*. Furthermore, where-
g 10. 10. as A.B *d* :: C.D; and inverſely A.C :: B.D :: C.B;
h 24. 10. and Bq is ρ̇ν. therefore ſhall CD (*k* Bq) be ρ̇ν.
k 17. 6. Laſtly, becauſe √ Aq − Bq *d* ⊓ A, *l* ſhall √ Cq
l 15. 10. − Dq be ⊓ C. therefore, &c. But if √ Aq − Bq
⊓ Aq, then ſhall √ Cq − Dq be ⊓ C.

In numbers; let there be A 8, B √ 48 (√ : 64 − 16) therefore C = √ AB = *v* √ 3072. and D = *v* √ 1728. wherefore CD = *v* √ 5308416 = √ 2304.

PROP. XXXIII.



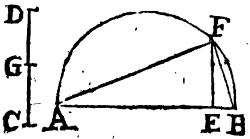
To find out two medial lines D, E, commenſurable in power only, comprehending a medial rectangle DE, ſo far that the greater D ſhall be more in power than the leſs E, by the ſquare of a right line commenſurable to the greater in length.

a Take A and C ρ̇ ⊤⊢, ſo that √Aq−Cq ⊓⊔ a 30. 10.
A. *b* take alſo B ⊤⊢ A and C, and make A. D b *lem.* 21.
c :: D. B *d* :: C. E. then D and E are the lines 10.
ſought for. c 13. 6.

For becauſe A and C *e* are ρ̇, *e* and B ⊤⊢ A d 12. 6.
and C, *f* therefore ſhall B be ρ̇, and D (√ AB) *g* e *conſtr.*
ſhall be μ. But becauſe A.D :: C.E, therefore in- f *ſch.* 12.
verſely A.C :: D. E. wherefore ſeeing A ⊤⊢ C, 10.
therefore D ſhall be ⊤⊢ E. therefore E is μ. g 22. 10.
Furthermore, *l* being D. B :: C. E. and BC is μν. h 10. 10.
alſo DE, equal to it, is μν. Laſtly, becauſe A. k 24. 10.
C :: D. E. *e* ſeeing √Aq − Cq ⊓⊔ A. therefore l 22. 10.
√ Dq − Eq ⊓⊔ D. therefore, &c. But if √Aq m 16. 6.
− Cq ⊓⊔ A. then √ Dq − Eq ⊓⊔ Eq. n 15. 5.

In numbers, let there be A 8, C √ 48. B √ 18. then D *v* √ 3072. and E *v* √ 588. wherefore D. E :: 2. √ 3. and DE = √ 1344.

PROP. XXXIV.



To find out two right
lines AF, BF, incommen-
ſurable in power, whoſe
ſquares added together
make a rational figure,
and the rectangle contain- a 31. 10.
ed under them medial. b 10. 1.

a Let there be found AB, CD, ρ̇ ⊤⊢ ; ſo that c 28. 6.
√ ABq − CDq ⊓⊔ AB. divide CD equally in d 12. 6.
G. *c* make the rectangle AEB = GCq. Upon e *cor.* 8. 6.
AB the diameter draw the ſemicircle AFB, e- & 17. 6.
rect the perpendicular EF, and draw AF, BF. f 7. 5.
theſe are the lines required. g 19. 10.

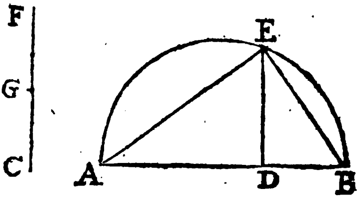
For AE. BE *d* :: BA × AE. AB × BE. But BA h 10. 10.
× AE *e* = AFq ; and AB × BE = FBq. *f* there- k 31. 3. &
fore AE. EB :: AFq. FBq. therefore being AE *g* 47. 1.
⊓⊔ EB, *h* AFq ſhall be ⊓⊔ FBq. Moreover ABq l *conſtr.*
(*k* AFq+FBq) *l* is ρ̇ν. Laſtly, EFq *l* = AEB *l* = m 1. *ax.* 1.
CGq. *m* therefore EF = CG. therefore CD × AB n 22. 10.
= 2 EF × AB. But CD × AB *n* is μν. *o* there- o 24. 10.
fore AB × EF, *p* or AF × FB is μν. *Which was* p *ſch.* 22. 6.
to be dem.

The Explication of the ſame by numbers.

Let AB be 6, CD $\sqrt{12}$. then CG $= \sqrt{\frac{12}{4}} = \sqrt{3}$. but AE $= 3 + \sqrt{6}$. and EB $= 3 - \sqrt{6}$. whence AF ſhall be $\sqrt{:18 + 216}$. and FB $\sqrt{18 - \sqrt{216}}$. Alſo AFq + FBq is 36, and AF x FB $= \sqrt{108}$.

But AE is found in this manner. Becauſe BA (6.) AF :: AF.AE. therefore 6 AE = AFq = AEq + 3 (EFq.) therefore 6 AE − AEq = 3. Put 3 + e = AE. then 18 + 6 e − 9 − 6 e − ee, that is, 9 − ee = 3. or ee = 6. wherefore e = $\sqrt{6}$ and ſo AE $= 3 + \sqrt{6}$.

PROP. XXXV.

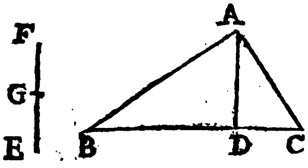


To find out two right liees AE, EB, incommenſuracle in power, whoſe ſquares added together make a medial figure, and the rectangle contained under them rational.

a 32. 10. *a* Take AB and CF μ ⊤⊥, ſo that AB x CF be ρν, and $\sqrt{ABq - CFq}$ ⊐⊏ AB. and let the reſt be done as in the prec. Prop. AE, EB are the line required.

For, as it is ſhewn there, AEq ⊐⊏ EBq. alſo
b *conſtr.* ABq (AEq + EBq) is μν, and laſtly, AB x CF
c *ſch.* 12. *b* is ρν, *c* therefore alſo AB x DE, that is, AE x
d *ſch.* 22. 6. EB, is ρν. therefore, &c.

PROP. XXXVI.



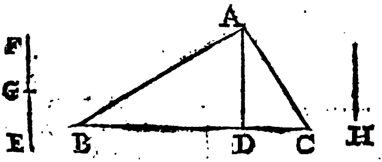
To find out two right lines BA, AC, incommenſurable in power, whoſe ſquares added together make a medial figure, and the rectangle alſo contained under them medial, and incommenſurable to the figure compoſed of the ſquares.

a Take BC and EF μ ⊤⊔, ſo that BC × EF be μν. and √ BCq − EFq ˥⊔ BC. and ſo forward, as in the prec. BA, AB, ſhall be the lines ſought for. a 33. 10.

For (as above) BAq ˥⊔ ACq. alſo BAq+ACq is μν. and BA × AC is μν. Laſtly, BC *b* ˥⊔ EF, and *c* ſo BC ˥⊔ EG; likewiſe BC.EG *d*:: BCq. BC × EG (BC × AD, or BA × AC) *e* therefore BCq. (ABq+ACq) ˥⊔ BA × AC. therefore, *&c.*

b *conſtr.*
c 13. 10.
d 1. 6.
e 14. 10.

Schol.



To find out two medial lines incommenſurable both in length and power.

a Take BC μ and let BA × AC be μν, and ˥⊔ BCq (BAq+ACq *b* make BA. H :: H. AC. then I ſay BC and H are μ ⊤⊔. For BC is μ. *a* and BA × AC (*c* Hq) is μν. wherefore H is alſo μ. *d* Likewiſe BA × AC ˥⊔ BCq; therefore Hq ˥⊔ BCq. therefore, *&c.*

a 36. 10.
b 33. 6.
c 17. 6.
d 14. 10.

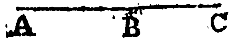
Here begin the ſenaries of lines irrational by compoſition.

PROP. XXXVII.

If two rational lines AB, BC, commenſurable only in power, be added together, the whole line AC is irrational, and is called a binomial line, or of two names.

a *hyp.* For becauſe AB *a* ˥⊏ BC, thence *b* ſhall ACq
b *lem.* 26. be ˥⊏ ABq. But AB *a* is a $\dot{\rho}$, *c* therefore AC
10. is $\dot{\rho}$. *Which was to be dem.*
c 11. *def.* 10.

PROP. XXXVIII.



If two medial lines AB, BC, only in power commenſurable, be compounded, and contain a rational rectangle, the whole line AC is irrational, and called a firſt bimedial line.

a *hyp.* For being that AB *a* ˥⊏ BC, *b* ſhall ACq be
b *lem.* 26. ˥⊏ AB × BC, $\dot{\rho}r$. *c* therefore AC is $\dot{\rho}$. *Which*
10. *was to be dem.*
c 11. *def.* 10.

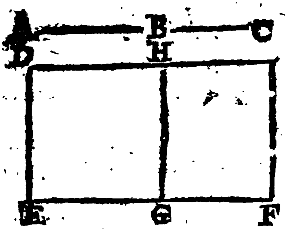
Lemma.

A rectangle A C, contained under a rational line AB and an irrational line BC, is irrational.

For if the rectangle AC be affirmed,
a *hyp.* $\dot{\rho}r$, *a* then being AB is $\dot{\rho}$, *b* the breadth BC
b 21. 10. ſhall be alſo $\dot{\rho}$. *againſt the Hyp.*

PROP.

PROP. XXXIX.



If two medial lines AB, BC, commensurable only in power, containing a medial rectangle, be compounded, the whole line AC ſhall be irrational, and is called a ſecond bimedial line.

Upon the propounded line DE $\mathring{\rho}$ *a* make the rectangle DF = ACq; *b* and DG = ABq + BCq. Becauſe ABq *c* ⊓⊔ BCq, *d* therefore ABq + BCq, *i. e,* DG, ⊓⊔ ABq: but ABq *e* is $\mu\nu$, *e* therefore DG is $\mu\nu$, but the rectangle ABC is taken $\mu\nu$; *e* and conſequently 2 ABC (*f* HF) is $\mu\nu$. *g* therefore EG and GF are $\mathring{\rho}$. Being alſo that DG *h* ⊓⊔ HF; and DG. HF :: *k* EG. GF; *l* therefore EG ⊓⊔ GF, *m* therefore the whole EF is $\mathring{\rho}$. *n* wherefore the rectangle DF is $\mathring{\rho}\nu$, *o* therefore $\surd$ DF, *i. e.* AC, is $\mathring{\rho}$. *Which was to be dem.*

a *cor.* 10. 6.
b 47. 1. & 12. 6.
c *hyp.*
d 16. 10.
e 24. 10.
f 4. 2.
g 27. 10.
h *lem.* 26. 10.
k 1. 6.
l 10. 10.
m 37. 10.
n *lem.* 38. 10.

PROP. XL.

If two right lines AB, BC commenſurable only in power, be added together, making that which is compoſed of their ſquares rational, and the rectangle contained under them medial, the whole right line AC is irrational, and is called a Major line.

For whereas ABq + CBq *a* is $\mathring{\rho}\nu$. and *b* ⊓⊔ 2 ABC *c* $\mu\nu$; and ſo ACq (*d* ABq + BCq + 2 ABC) *e* ⊓⊔ ABq + BCq $\mathring{\rho}\nu$, *f* therefore ſhall AC be $\mathring{\rho}$. *Which was to be dem.*

a *hyp.*
b *ſch.* 12. 10.
c *hyp.* and 24. 10.
d 4. 2.
e 17. 10.
f 11. *def.* 10.

PROP.

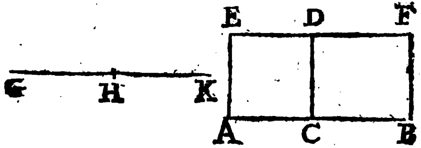
PROP. XLI.

If two right lines AC, CB, incommenſurable in power, be added together, having that which is made of their ſquares added together medial, and the rectangle contained under them rational, the whole right line AB ſhall be irrational; and is called A line containing in power a rational and a medial rectangle.

a *hyp*, and *ſch.* 12.10.
b *ſch.* 12. 10.
c *hyp.*
d 17. 10.
e 11. *def.* 10.

For 2 rectangles ACB *a* ῥν, *b* ⊔ ACq + CBq *c* μν. *d* therefore 2 ACB *d* ⊔ ABq. wherefore *e* AB is ῥ. *Which was to be dem.*

PROP. XLII.



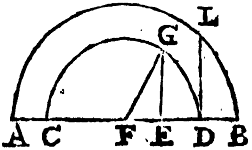
If two right lines GH, HK, incommenſurable in power be added together, having both that which is compoſed of their ſquares medial, and the rectangle contained under them medial, and incommenſurable to that which is compoſed of their ſquares, the whole right line GK is irrational, and is called a line containing in power two medial figures.

a *hyp.*
b 23. 10.
c 4. 2.
d 2. 6.
e 20. 10.
f 37. 10.
g *lem.* 38. 10.
h 21. *def.* 10.

Upon the propounded line FB ῥ make the rectangles AF=GKq, and CF=GHq+HKq. Being GHq+HKq (CF) *a* is μν, the breadth CB *b* ſhall be ῥ. Alſo becauſe 2 rectangles GHK (*c* AD) *a* is μν, therefore AC *b* ſhall be ῥ. Moreover becauſe the rectangle AD *a* ⊔ CF, *d* and AD.CF :: AC. CB. *e* thence ſhall *e* AC be ⊔ CB. *f* wherefore AB is *g* ῥ. therefore the rectangle AF. *i. e.* GKq is ῥν; *h* and conſequently GK is ῥ. *Which was to be dem.*

PROP.

PROP. XLIII.

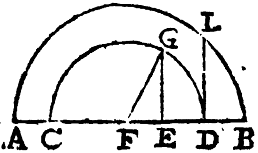


A line of two names, or binomial, AB, can at one point only D be divided into its names, AD, DB.

If it be poſſible, let the binomial line AB be divided at the point E, into other names AE, EB. It is manifeſt that the line AB is in both caſes divided unequally, ſince AD ⊏ DB, and AE ⊏ EB.

Becauſe the rectangles ADB, AEB *a* are *μα*; a 37. 10.
ν and each of ADq, DBq, AEq, EBq is *ῥα*. *b* and b *ſch.* 27.
ſo ADq + DBq *b* and AEq + EBq are alſo *ῥα*. *b* there- 10.
fore ADq + DBq −: AEq + EBq *c i. e.* 2 AEB − 2 c *ſch.* 5. 2.
ADB is *ῥν*. *d* therefore AEB − ADB is *ῥν*. therefore d *ſch.* 12.
μν exceeds *μν* by *ῥν*. *e Which is abſurd.* 10.
e 27. 10.

PROP. XLIV.

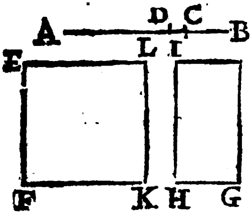


A firſt bimedial line AB, is in one point only D divided into its names AD, DB.

Conceive AB to be divided into other names AE, EB, whereupon every one ADq, DBq, EBq, a 38. 10.
will be *a μα*. and the rectangles ADB, AEB, b *ſch.* 27.
and the doubles of them *ῥα*. *b* therefore 2 AEB 10.
− 2 ADB. *c i. e.* ADq + DBq −: AEq + EBq c *ſch.* 5. 2.
is *ῥν*. *Which is abſurd.* d 27. 10.

P PROP.

PROP. XLV.

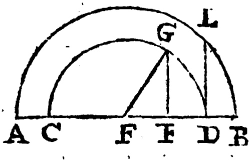


A ſecond bimedial line AB, is divided into its names AC, CB, only at one point C.

Suppoſe there were other names AD, DB. Upon the propounded line EF ῥ make the rectangles EG = ABq, and EH = ACq + CBq. as alſo EK = ADq + DBq.

a 39. 10. Becauſe ACq, BCq *a* are μα ⊏. ; *b* ACq +
b 16. *and* CBq (EH) ſhall be μν. *c* therefore the breadth
24. 10. FH is ῥ. *a* moreover the rectangle ACB, *d* and ſo
c 23. 10. 2 ACB (*e* IG) is μν. *c* therefore HG is alſo ῥ.
d 24. 10. And ſince EH is *f* ⊏⊐ IG, *g* and EH.IG :: FH.
e 4. 2. HG. *h* therefore FH, HG ſhall be ⊏⊐. *k* there-
f *lem.* 26. fore FG is a binomial, whoſe names are FH,
10. HG. By the ſame reaſon FG is binomial, and
g 1. 6. the names of it FK, KG : *contrary to the* 43. *of*
h 10. 10. *this Book.*
k 37. 10.

PROP. XLVI.

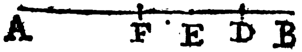


A Major line AB is at one point only D divided into its names, AD, DB.

Imagine other names AE, EB. whereupon the
a 40. 10. rectangles ADB, AEB, are *a* μα. *a* and as well
b *ſch.* 27. ADq + DBq, as AEq + EBq are ῥα. *b* therefore
10. ADq + DBq —: AEq + EBq, *c i.e.* 2 AEB —
c *ſch.* 5. 2. 2 ADB is ῥν. *Which is impoſſible.*
d 27. 10.

PROP.

PROP. XLVII.

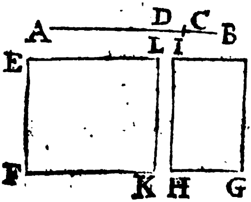


A line AB containing in power a rational and a medial figure is divided at one point only D into its names AD, DB.

Conceive other names AE, EB. then both AEq + EBq, and ADq + DBq are μα. *a* and the rectangles AEB, ADB are ῥα. *b* therefore 2 AEB − 2 ADB, *c i. e.* ADq + DBq −: AEq + EBq is ῥv. *d Which is abſurd.*

a 41. 10.
b ſch. 17. 10.
c ſch. 5. 2.
d 27. 10.

PROP. XLVIII.



A line AB containing in power two medial rectangles, is at one point only C divided into its names AC, CB.

If you will divide AB into other names AD, DB. draw upon the line propounded EF ῥ the rectangles EG = ABq, and EH = ACq + CBq, and EK = ADq + DBq. then becauſe ACq + CBq, namely, EH, *a* is μυ, *b* the breadth FH ſhall be ῥ. Alſo becauſe 2 ACB, *c* that is, IG, is *a* μυ, HG *b* ſhall be likewiſe ῥ. Therefore, whereas EH *a* ⊐⊏ IG. and EH. IG *d* :: FH. HG, thence FH *e* ſhall be ⊐⊏ HG. *f* therefore FG is a binomial, and the names of it FH, HG. In like manner FK, KG ſhall be the names of it, *againſt the* 43. *of this Book.*

a 42. 10.
b 23. 10.
c 4. 2.
d 1. 6.
e 10. 10.
f 37. 10.

Second Definitions.

A Rational line being propounded, and the binomial divided into its names, the greateſt of whoſe names is more in power than the leſs by the ſquare of a right line commenſurable to the greater in length; then

I. If the greater name be commenſurable in length to the rational line propounded, the whole line is called a firſt binomial line.

II. But if the leſſer name be commenſurable in length to the rational line propounded, the whole line is called a ſecond binomial.

III. If neither of the names be commenſurable in length to the rational line propounded, it is called a third binomial.

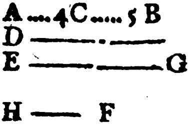
Furthermore if the greater name be more in power than the leſs, by the ſquare of a right line incommenſurable to the greater in length, then

IV. If the greater name be commenſurable to the propounded rational line in length, it is called a fourth binomial.

V. If the leſſer name be ſo, a fifth.

VI. If neither, a ſixth.

PROP. XLIX.



To find out a firſt binomial line, EG.

a ſch. 29. 10. *a* Take AB, AC, ſquare numbers, whoſe exceſs CB is not Q. let D be propounded ρ̇. *b* Take EF ⊓ D, and *c* make AB. CB :: EFq, FGq. then EG ſhall be a 1. bin.

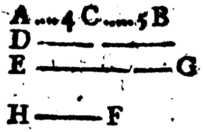
b 2. lem. 10. 10.
c 3. lem. 10. 10.
d conſtr.
e 6.def.10.
f 6. 10.
g ſch. 12. 10.
h 9. 10.
k 9. 10.
l 1.def.48. 10.

For EF *d* ⊓ D. *e* therefore EF is ρ̇. *f* alſo EFq ⊓ FGq. *g* therefore FG is alſo ρ̇. likewiſe *d* be EFq. FGq :: AB.CB :: Q. not Q. *h* therefore EF ⊓ FG. Laſtly, becauſe by converſion of proportion, EFq, EFq − FGq :: AB.AC :: Q.Q. thence EF *k* ſhall be ⊓ √ EFq — FGq. *l* therefore EG is a firſt binomial. *Which was to be done.*

In numbers thus; let there be D 8. EF 6. AB 9. CB 5. wherefore becauſe 9. 5 :: 36. 20, therefore FG is √ 20. and conſequently EG is 6 + √ 20.

PROP.

PROP. L.



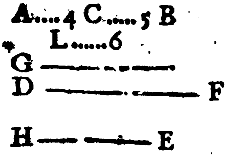
To find out a ſecond binomial line, EG.

Take AB and AC ſquare numbers, the exceſs of which is CB not Q. Let D be the line propounded ρ̇. Prove it as the prec.
take FG ⊓⊔ D, and make CB. AB :: FGq. EFq.
then EG will be the line deſired.

For FG ⊓⊔ D. wherefore FG is ρ̇. Alſo EFq
⊓⊔ FGq. therefore EF is ρ̇. Likewiſe becauſe
FGq. EFq :: CB, AB :: not Q. Q. thence FG is
⊓⊔ EF. Laſtly, ſeeing CB. AB :: FGq.EFq. and
inverſely AB. CB :: EFq.FGq. therefore as in the
foregoing Prop. EF ⊓⊔ √ EFq − FGq. *a* where- a 2.def.48.
by EG is a 2.binomial. *Which was to be done.* 10.

In numbers; let there be D 8, FG 10, AB 9, CB 5, then EF is √ 180. wherefore EG is 10 + √ 180.

PROP. LI.



To find out a third binomial line, DF.

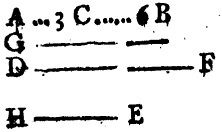
a Take A B, A C, a ſch. 19.
ſquare numbers, the 10.
exceſs of which B C is not Q. and let L be a number not Q next greater than CB, *viz.* by a unit or two. Let G
be the line propounded ρ̇. *b* Make L. AB :: Gq,
DEq. *b* and AB. CB :: DEq.EFq. then DF ſhall b 3. lem.
be a 3. bin. 10. 10.

For becauſe DEq *c* ⊓⊔ Gq, *d* DE is ρ̇. alſo Gq. c conſtr. 6.
DEq :: L. AB :: not Q. Q. *e* therefore G ⊓⊔ DE. 10.
Likewiſe being that DEq *e* ⊓⊔ EFq, *d* alſo EF is d ſch. 12.
ρ̇. Moreover becauſe DEq. EFq :: AB. CB :: Q. not 10.
Q. *f* is DE ⊓⊔ EF. and being that by conſtr. and e 6. 10.

f 9. 10. of equality Gq. EFq :: L. CB :: not Q. Q. (for
g ſch. 27. 8. g L and CB are not like plane numbers) h there-
h 9. 10. fore ſhall G be alſo ⊔ EF. Laſtly, as in the
prec. Prop. √ DEq − EFq ⊔ DE. k therefore
k 3. def. 48. 10. DF is a 3 bin. *Which was to be done.*

In numbers; let there be AB, 9. CB, 5. L, 6. G, 8. then ſhall be DE √ 96, and EF $\sqrt{\frac{480}{9}}$. wherefore DF = √ 96 + $\sqrt{\frac{480}{9}}$.

PROP. LII.



To find out a fourth binomial line, DF.

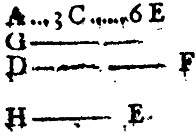
a ſch. 29. 10. a Take any ſquare number AB, and divide it into AC, CB not ſquares. Let G be the

b 2. lem. 10. line propounded ρ. b take DE ⊔ G. c and make AB. CB :: DEq. EFq. then DF ſhall be a 4 bin.

c 3. lem. 10. 10. For, as in the 49 of this Book, DF may be ſhewn to be a binom. and alſo becauſe by conſtr.
d 9. 10. and converſion of proportion DEq. DEq − EFq
e 4. def. 48. 10. :: AB. AC :: Q not Q. d ſhall DE be ⊔ √ DEq − EFq. e therefore DF is a 4 bin.

In numbers, let G be 8, DE, 6. then EF ſhall be √ 24. therefore DF is 6 + √ 24.

PROP. LIII.



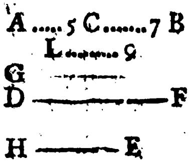
To find out a fifth binomial line, DF.

Take any ſquare number AB, whoſe ſegments AC, CB are not Q. Let G be the line propounded ρ. take EF ⊔ G. and make CB. AB :: EFq. DEq. then ſhall DF be a 5 bin.

For DF ſhall be a bin. as in the 50. of this Book, and becauſe by conſtruction, and inverſion DEq. EFq :: AB. CB. and ſo by converſion of proportion, DEq. DEq − EFq :: AB. AC :: Q. not Q. *a* therefore ſhall DE be ⊓⊔ √ DEq − EFq. *b* therefore DF is the 5 bin. *Which was to be done.* a 9. 10. b 5. *def.* 48. 10.

In numbers, let there be G,7. EF,6. then DE ſhall be √ 54. wherefore DF is 6 + √ 54.

PROP. LIV.



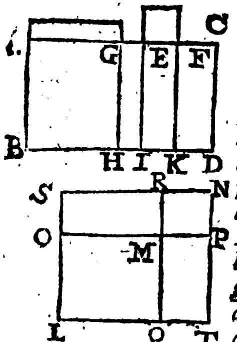
To find out a ſixth binomial line.

Take AC;CB, prime numbers, ſo that AC + CB (AB) be not Q. take alſo any number ſquare L. Let G be the line propounded ρ. *a* and make L. AB :: Gq. DEq. and AB. CB :: DEq. EFq. then DF ſhall be a 6 binomial. a 3. *lem.* 10. 10.

For DF may be demonſtrated bin. as in the 51. of this Book, and alſo by reaſon that DE and EF ⊓⊔ G. laſtly, likewiſe becauſe by conſtr. and converſion of proportion DEq. DEq − EFq :: AB. AC :: not Q. Q. (For AB is prime to AC, *b* and ſo unlike to it) *c* therefore DE ⊓⊔ √ DEq − EFq. *d* therefore DF is a 6 bin. *Which was required.* b *ſch.* 27. 8. c 9. 10. d 6. *def.* 48. 10.

In numbers, let there be G 6. DE √ 48. then EF ſhall be √ 28. wherefore DF is √ 48 + √ 28.

Lemma.



a 28. 6.
b 31. 1.
c 14. 2.

Let AD be a rectangle, and the ſides thereof AC divided unequally in E; alſo let the leſſer portion EC be equally divided in F, upon the line AE a *make the rectangle AGE = EFq. and from the points G, E, F* b *draw GH, EI, FK, parallel to AB.* c *Let the ſquare LM be made equal to the rectangle AH, and upon OMP produced the ſquare MN = GI, and let the right lines LOS, LQT, NRS, NPT be produced.*

I ſay 1. MS, MT, *are rectangles.* For by reaſon of the right angles of the ſquares OMQ, RMP,
a *ſch.*15.1. *a* ſhall QMR be a right line. *b* therefore RMO,
b 13. 2. QEP, are right angles. wherefore the parallelograms MS, MT are rectangles.

c 2.*ax.*21. 2. *Hence it is plain that LS c = LT, and conſequently that LN is a ſquare.*

3. *The rectangles SM, MT, EK, FD are equal.*
d *hyp.* For becauſe the rectangle AGE *d* = EFq. *e*
e 17. 6. thence ſhall AE. EF :: EF. GE. *f* and ſo AH. EK
f 1. 6. :: EK. GI. that is by conſtr. LM. EK :: EK. MN.
g *ſch.*22.6. *g* but LM. SM :: SM, MN. therefore EK *h* = SM
h 9. 5. *k* = FD *l* = MT.

k 36. 1. 3. *Hence LN m = AD.*

l 43. 1. 5. *Being that EC is equally divided in F, it is*
m 2. *ax.*1. *plain that EF, FC, EC are* ⊓⊔.

n 16. 10. 6. If AE ⊓̶ EC, and AE ⊓⊔ √ AEq − ECq,
o 18. and *o* then ſhall AG, GE, AE, be ⊓⊔ alſo, becauſe
16. 10. AG. GE :: AH. GI. *p* therefore ſhall AH, GI,
p 10. 10. *i. e.* LM, MN, be ⊓⊒. Likewiſe thereupon,

7. OM ⊔ MP. For by the Hyp. AE ⊔ EC. *q* therefore BC ⊔ GE. *q* wherefore EF ⊔ q 14. 10.
GE. but EF. GE :: EK. GI. *r* therefore EK ⊔ r 10. 10.
GI, that is, SM ⊔ MN. but SM. MN :: OM. MP. *r* therefore OM ⊔ MP.

8. If AE be ſuppoſed ⊔ √ AEq − ECq, it is *s* 19. *and*
apparent that AG, GE, AE, are ⊔. whence 17. 10.
LM ⊔ MN. for AG.GE :: AH.GI :: LM.MN.

Theſe being well conſidered, we ſhall eaſily diſpatch the ſix following Propoſitions.

PROP. LV.

If a ſpace AD be contained under a rational line AB, and a firſt binomial line AC (AE + EC) the right line OP which containeth that ſpace in power is irrational, and called a binomial line.

All that being ſuppoſed which is deſcribed and demonſtrated in the next foregoing *Lemma*, it is manifeſt that the right line OP containeth in power the ſpace AD. *a* Likewiſe AG, GE, a *hyp. and*
AE are ⊔. therefore ſeeing AE is ῥ ⊔ AB. *lem.* 54. 10.
c ſhall alſo AG and GE be ῥ ⊔ AB. *d* there- b *hyp.*
fore the rectangles AH, GI, that is the ſquares c *ſch.* 12.
LM. MN are ῥα. therefore OM, MP are ῥ *e* ⊓. 10.
f and conſequently OP is a binomial. *Which* d 20. 10.
was to be dem. e *lem.* 54.

In numbers, let there be AB 5. AC 4 + √ 12. 10.
wherefore the rectangle AD = 20 + √ 300 = f 37. 10.
to the ſquare LN. therefore OP is √ 15 + √ 5. namely a 6 binomial.

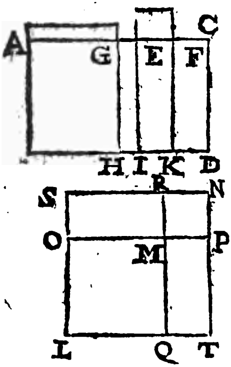
PROP. LVI.

If a ſpace AD be comprehended under a rational line AB, and a ſecond binomial AC (AE + EC) the right line OP, which contains that ſpace AD in power, is irrational, and called a firſt medial line.

The foreſaid *Lemma* of the 54 of this Book be-
a *hyp.* and ing again ſuppoſed, then ſhall OP be = √ AD
lem. 54. 10. *a* alſo AE, AG, GE are ⊤⊥ therefore ſince AE *b*
b *hyp.* is ῥ ⊤⊔ AB, likewiſe AG, GE *c* ſhall be ῥ ⊤⊔
c *ſch.* 12. AB. therefore the rectangle AH, GI, *i. e.* OMq,
10. MPq. *d* are *μα.* *e* Moreover OM ⊤⊥ MP. Laſt-
e *lem.* 54. ly, EF ⊤⊔ EC, and EC *f* ⊤⊔ AB, *g* wherefore
10. EK. *i. e.* SM, or OMP is ῥv. *h* Conſequently OP
f *hyp.* 12. is a firſt bimedial. *Which was to be dem.*
10. In numbers, let there be AB, 5; and AC √
g 20. 10. 48: + 6. then the rectangle AD = √: 1200 + 30
h 38. 10. = OPq. therefore OP is *v* √ 675 + *v* √ 75. *viz.*
a firſt bimedial.

See Scheme 57.

PROP. LVII.

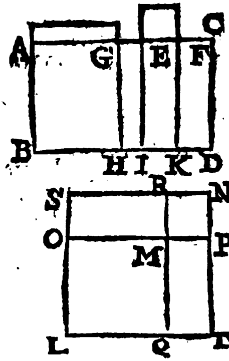


If a ſpace AD be contained under a rational line AB and a third binomial line AC (AE + EC) the right line OP which contains in power the ſpace AD, is irrational, and called a ſecond bimedial line.

As above, OPq = AD. alſo the rectangles AH, GI, that is OMP, MPq
a *hyp.* and 22. 10. are *μα.* *a* Likewiſe EK or OMP is *μν.*
b 39. 10. *b* therefore OP is a ſecond bimedial.

In numbers; let there be AB 5. AC √ 32 + √ 34. wherefore AD is √ 800 + √ 600 = OPq. and ſo OP is *v* √ 450 + *v* √ 50. that is a 2. bimed.

PROP. LVIII.



If a ſpace AD be comprehended under a rational line AB and a fourth binomial AC (AE + EC) the right line OP containing the ſpace AD in power, is that irrational line which is called a Major line.

For again, OMq *a* ⊔ a lem. 54.
MPq; and the rectangle 10.
AI, *i. e.* OMq + MPq *b* b hyp. and
is *ῥν.* *c* alſo EK or OMP 20. 10.
is *μν.* *d* therefore OP (√ c hyp. and
AD) is a Major line. 22. 10.
Which was to be dem. d 40. 10.

In numbers; let there be AB 5. and AC 4 + √8. then the rectangle AD is 20 + √200. wherefore OP is √: 20 + √200.

PROP. LIX.

If a ſpace AD be contained under a rational line AB, and a fifth binomial AC, the right line OP which contains the ſpace AD in power, is that irrational line, which is a line containing a rational and a medial rectangle in power.

Again OMq ⊔ MPq. and the rectangle AI
or OMq + MPq is *μν.* *a* Likewiſe the rectangle a *As in the*
EK or OMP is *ῥν,* *b* therefore OP (√ AD) con- *prec.*
tains in power *ῥν* and *μν.* *Which was to be dem.* b 41. 10.

In numbers, let there be AB 5. and AC 2 + √8. then the rectangle AD = 10 + √200 = OPq. wherefore OP is √: 10 + 200.

PROP.

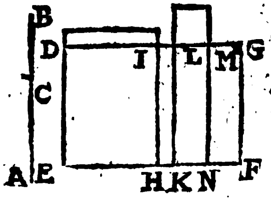
PROP. LX.

If a ſpace AD be contained under a rational line AB and a ſixth binomial AC (AE+EC) the line OP containing the ſpace AD in power is irrational, which contains in power two medial rectangles.

As often before, OMq ⊐⊏ MPq, and OMq + MPq is μν. and alſo the rectangle (EK) OMP is μν. *a* therefore OP = √ AD contains in power 2 μν. *Which was to be dem.* — a 42. 10.

In numbers, let there be AB 5. AC √ 12 + √ 8. therefore the rectangle AD or OPq is √ 300 + √ 200. and ſo OP is √: √ 300 + √ 200.

Lemma.



Let a right line AB be unequally divided in C, and let AC be the greater portion, and upon ſome line DE apply the rectangles DF = ABq, and DH = ACq, and IK = CBq. and let LG be divided equally in M, and alſo MN drawn parallel to GF.

a 4.2. *and* 3. *ax.* 1. — I ſay 1. The rectangle ACB is = LN or MF. *a* For 2 ACB = LF.

b 7. 2. — 2. DL ⊏ LG. for DK (ACq + CBq) *b* ⊏ LF
c 1. 6. — (2 ACB) therefore being DK, LF are of equal
d 16. 10. — altitude, *c* DL ſhall be ⊏ LG.

3. If AC ⊐ CB, *d* then ſhall the rectangle DK be ⊐⊏ ACq and CBq.

e *lem.* 26. 10. — 4. *Alſo* DL ⊐⊏ LG. for ACq + CBq *e* ⊐⊏ 2 ACB, *i. e.* DK ⊐⊏ LF. but DK. LF *e* :: DL.
f 10. 10. — LG. *f* therefore DL ⊐⊏ LG.

g 1. 6. — 5. *Moreover* DL ⊐⊏ √ DLq − LGq. For
h 17. 6. — ACq. ACB *g* :: ACB. CBq. that is, DH. LN ::
k *hyp.* — LN. IK. *e* wherefore DI. LM :: LM. IL. *h*
l 10. 10. — therefore DI × IL = LMq. therefore ſeeing ACq
m 18. 10. — *k* ⊐⊏ CBq, that is, DH ⊐⊏ IK. and *l* to DI ⊐⊏ IL. *m* ſhall DL be ⊐⊏ √ DLq − LGq. *Which was to be dem.*

6. But if ACq be put ⊐⊏ CB, n then ſhall DL be ⊐⊏ √DLq − LGq. n 19. 10.

This Lemma is preparatory to the following Propoſitions.

PROP. LXI.

The ſquare of a binomial line (AC + CB) applied unto a rational line DE, makes the breadth DG a firſt binomial line.

Thoſe things being ſuppoſed, which are deſcribed and demonſtrated in the next preceding *Lemma*; becauſe AC, CB *a* are ρ̇ ⊐⊢, *b* the rectangle DK ſhall be ⊐⊏ ACq. *c* and ſo DK is ρ̇ν. *d* therefore DL ⊐⊏ DE ρ̇. but the rectangle ACB, and ſo 2 ACB (LF) *e* is μν. *f* therefore the latitude LG is ρ̇ ⊐⊏ DE. *g* therefore alſo DL ⊐⊏ LG alſo DL ⊐⊏ √DLq − LGq. from whence *k* it follows that DG is a firſt binomial. *Which was to be dem.*

a *hyp.*
b *lem.* 60. 10.
c *ſch.* 12.
d 21. 10.
e 22. *and* 24. 10.
f 23. 10.
g 13. 10.
h *lem.* 60. 10.
k 1. *def.* 10. 10.

PROP. LXII.

The ſquare of a firſt bimedial line (AC + CB) being applied to a rational line DE, makes the breadth DG a ſecond binomial line.

The aforeſaid *Lemma* being again ſuppoſed; the rectangle DK ⊐⊏ ACq. *a* therefore DK is μν. *b* therefore the breadth DL is ρ̇ ⊐⊏ DE. But becauſe the rectangle ACB, and ſo LF (2 ACB) *c* is ρ̇ν, *d* ſhall LG be ρ̇ ⊐⊏ DE. *e* therefore DL, LG are ⊐⊏ *f* alſo DL ⊐⊏ √DLq − LGq. *g* from whence it is clear that DG is a ſecond binomial. *Which was to be dem.*

a 24. 10.
b 23. 10.
c *hyp. and ſch.* 22. 10.
d 2. 10.
e 13. 10.
f *lem.* 10. 10.
g 2. *def.* 48. 10.

PROP. LXIII.

The ſquare of a ſecond bimedial line (AC+CB) applied to a rational line DE makes the breadth DG a third binomial line.

a *hyp.* *and* As in the prec. DL is ρ̇ ⌐⊏ DE. Further-
24. 10. more becauſe the rectangle ACB, and ſo LF (2
b 23. 10. ACB) *a* is μν. *b* therefore ſhall LG be ρ̇ ⌐⊏ DE.
c *lem.* 60. *c* Moreover DL ⌐⊏ LG. and alſo DL ⌐⊏ √
10. DLq — LGq. *d* therefore DG is a third bino-
d 3. *def.* 48. mial. *Which was to be dem.*
10.

PROP. LXIV.

The ſquare of a Major line (AC+CB) applied to a rational line DE, makes the breadth DG a fourth binomial line.

a *hyp.* *and* Again ACq+CBq. *i. e.* DK *a* is ρ̇ν *b* therefore
ſch. 12. 10. DL is ρ̇ ⌐⊏ DE, alſo ACB, and ſo LF (2 ACB)
b 21. 10. *c* is μν. *d* therefore LG is ρ̇ ⌐⊏ DE, *e* and con-
c *hyp.* *and* ſequently DL ⌐⊏ LG. Laſtly, becauſe AC ⌐⊑
24. 10. BC. *f* ſhall DL be ⌐⊏ DLq - LGq. *g* whence DG
d 23. 10. is a fourth binomial. *Which was to be dem.*
e 13. 10.
f *lem.* 60.
10.

PROP. LXV.

g 4. *def.* 48.
10.

The ſquare of a line containing in power a rational and a medial rectangle (AC -|- CB) applied to a rational line DE makes the breadth DG a fifth binomial.

a 23. 10. Again, DK is μν. *a* therefore DL is ρ̇ ⌐⊏ DE.
b 21. 10. alſo LF is ρ̇ν. *b* therefore LG is ρ̇ ⌐⊏ DE. *c*
c 13. 10. therefore DL ⌐⊏ LG. *d* likewiſe DL ⌐⊏ √
d *lem.* 60. DLq—LGq. *e* and ſo by conſequence DG is a
10. fifth binomial. *Which was to be dem.*
e 5. *def.* 48.

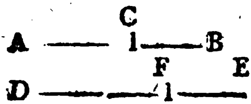
PROP. LXVI.

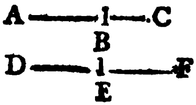
The ſquare of a line containing in power two medial rectangles (AC+CB) applied to a rational line DE, makes the breadth DG a ſixth binomial line.

As before, DL and LG are ⊔̸ DE. But *a* for that ACq + CBq (DK) *a* ⊔ ACB, *b* and so DK ⊔ LF (2 ACB) and alſo DK. LF *c* :: DL. LG. *d* therefore ſhall DL be ⊔ LG. *e* Laſtly, DL ⊔ √ DLq − LGq. *f* by which it appears that DG is a ſixth binomial.

a *hyp.*
b 14. 10.
c 1. 6.
d 10. 10.
e *lem.* 60. 10.
f 6. *def.* 48. 10.

Lemma.





Let AB, DE be ⊔. and make AB. DE :: AC. DF.

I ſay 1. AC ⊔ DF. as appears by 10. 10. alſo CB ⊔ FE. *a* becauſe AB. DE :: CB. FE. — a 19. 5.

2. AC. CB :: DF. FE. For AC. DF :: AB. DE :: CB. FE. therefore inverſely AC. CB :: DF. FE.

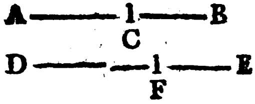
3. *The rectangle* ACB ⊔ DFE. For ACq. ACB *b* :: AC. CB *c* :: DF. EF :: DFq. DFE. wherefore by inverſion ACq. DFq :: ACB. DFE. therefore being ACq ⊔ DFq. *d* ſhall ACB be ⊔ DFE. — b 1. 6. c *before* d 10. 10.

4. ACq + CBq ⊔ DFq + EFq. For becauſe ACq. CBq *e* :: DFq. FEq. therefore by addition ACq + CBq. CBq :: DFq + FEq. FEq. therefore being CBq ⊔ FEq, *f* ſhall alſo ACq + CBq be ⊔ DFq + FEq. — e 22. 6. f 10. 10.

5. *Hence*, If AC ⊔̸ or ⊔̸ CB. *g* then likewiſe ſhall DE be ⊔̸ or ⊔̸ EF. — g 10. 10.

PROB.

PROP. LXVII.



A line DE, commenſurable in length to a binomial line (AC+CB) is it ſelf a binomial line, and of the ſame order.

Make AB. DE :: AC. DF. *a* then are AC, DF
a *lem.* 66. ⊐⊏ *a* and CB, FE '⊐⊏. whence being that AC
10. and CB *b* are ρ̇ ⊐⊤, *c* thence DF, FE ρ̇ ⊐⊤ there-
b *hyp.* fore DE is a binomial. But for that AC. CB *a* ::
c *lem.* 66. DF. FE. if AC ⊐⊏ or '⊐⊏ √ ACq − BCq. *d*
10. *and ſch.* then in like manner DF ⊐⊏ or '⊐⊏ √ DFq −
12. 10. FEq. alſo if AC ⊐⊏ or '⊐⊏ ρ̇ propounded, *e*
d 15. 10. then ſhall DF be ⊐⊏ or '⊐⊏ ρ̇ propounded. But
e 12. 10. if CB ⊐⊏ or '⊐⊏ ρ̇. likewiſe FE ⊐⊏ or '⊐⊏ ρ̇.
and 14. 10. If both AC, CB, '⊐⊏ ρ̇. *g* that is, whatſoever
f *by def.* 48. binomial AB is, DE ſhall be of the ſame order.
10. *Which was to be dem.*
g 14. 10.

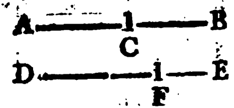
PROP. LXVIII.

A line DE commenſurable in length to a bimedial line (AC+CB) is alſo a bimedial line, and of the ſame order.

a 12. 6. Make AB. DE :: AC. DF. *b* therefore AC ⊐⊏
b *lem.* 66. DF and CB ⊐⊏ FE. therefore ſeeing AC and
10. CB *c* are μ, *d* alſo DF, and FE ſhall be μ, and
c *hyp.* for that AC *e* ⊐⊤ CB. *e* therefore FD ⊐⊤ FE.
d 24. 10. *f* therefore DE is 2 μ. Wherefore if the rectan-
e 10. 10. gle ACB be ρ̇ν. becauſe DFE *b* ⊐⊏ ACB, *g* like-
f 38. 10. wiſe DFE is ρ̇ν. and if that be μν. *h* this ſhall
g *ſch.* 12. be μν too. *k* That is, whether AB be 1 bimed.
10. or 2 bimed. DF ſhall be of the ſame order
k 38, *or* 39 *Which was to be dem.*
10.

PROP.

PROP. LXIX.



A line DE commenſurable to a Major line (AC + CB) is it ſelf a Major line.

Make AB. DE :: AC. DF. Becauſe AC *a* ⊤⊐ CB, *b* thence DF ⊤⊐ FE. Alſo ACq+CBq *a* is *ρ̇ν*. and ſo being DFq +FEq *b* ⊓ ACq+CBq, *c* alſo DFq+FEq is *ρ̇ν*. laſtly, the rectangle ACB *a* is *μν*. *d* therefore the rectangle DFE is *μν*. (becauſe DFE is *b* ⊓ ACB) *e* wherefore DE is a Major line.

a *hyp.*
b *lem.* 66. 10.
c *ſch.* 12. 10.
d 24. 10.
e 40. 10.

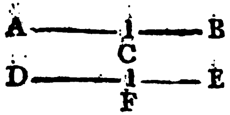
PROP. LXX,

A line DE commenſurable to a line containing in power a rational and a medial rectangle (AC + CB) is a line containing in power a rational and a medial rectangle.

Again make AB. DE :: AC. DF. Becauſe AC *a* ⊤⊐ CB, *b* alſo DF ⊤⊐ FE. likewiſe becauſe ACq+CBq *a* is *μν*, *c* therefore DFq+FEq ſhall be *μν*. laſtly, becauſe the rectangle ACB *a* is *ρ̇ν*, *d* alſo DFE is *ρ̇ν*. Therefore DE contains in power *ρ̇ν* and *μν*. *Which was to be dem.*

a *hyp.*
b *lem.* 66. 10.
c 24. 10.
d *ſch.* 12. 10.
e 41. 10.

PROP. LXXI.



A line DE commenſurable to a line containing two medial rectangles in power (AC + CB) is alſo a line containing in power two medial rectangles.

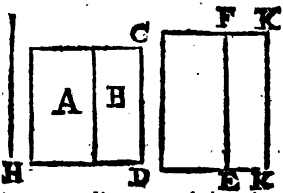
Divide DE, as in the prec. Becauſe ACq *a* ⊓ CBq, *b* thence ſhall DFq be ⊓ FEq, alſo for that ACq+CBq *a* is *μν*, *c* ſhall DFq+FEq be alſo *μν*. And in like manner becauſe ACB *a* is *μν*. *d* alſo DFE is *μν*. Laſtly, becauſe ACq+CBq ⊓ ACB,

a *hyp.*
b *lem.* 66. 10.
c 24. 10.
d 24. 10.

c ſhall

e 14. 10. *e* ſhall DFq + FEq be ⊓⊔ DFE. *f* from whence
f 42. 10. it follows that DE contains in power 2 μν.
Which was to be dem.

PROP. LXXII.

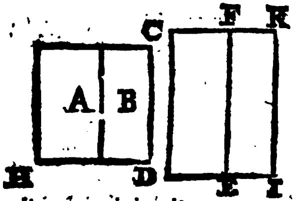


If a rational rectangle A, and a medial B, be compoſed together, theſe four irrational lines will be made; either a binomial, or a firſt bimedial, or a major, or a line containing in power a rational and a medial rectangle.

Namely, if Hq = A + B. then H ſhall be one of
the four lines which the Theorem mentions. For
a *cor.* 16.6. upon CD the propounded ῥ, *a* make the rectangle
b 2. *ax.* 1. gle CE = A, and FI = B. *b* and ſo CI = Hq.
c 21. 10. Whereas then is A ῥν, likewiſe CE is ῥν. *c* there-
d 23. 10. fore the latitude CF is ῥ ⊓⊔ CD. and becauſe B
e 13. 10. is μν, alſo FI ſhall be μν. *d* therefore FK is ῥ ⊓⊔
f 37. 10. CD. *e* therefore CF, FK are ῥ ⊓⊔. and ſo the
g 1. 6. whole CK *f* is binom. wherefore if A ⊏ B, *i. e.*
h 1. *def.* 48. CE ⊏ FI, *g* then CF ⊏ FK. therefore if CF ⊓⊔
10. √ CFq − FKq, *h* likewiſe CK ſhall be a 1 bin.
k 55. 10. and conſequently H = √ CI *k* is a bin. If CF
l 4. *def.* 48. be ſuppoſed ⊓⊔ √ CFq − FKq, *l* then ſhall CK
10. be a 4 bin. wherefore H (√ CI) *m* is a major
m 58. 10. line. But if A ⊐ B, *g* then ſhall CF be ⊐ FK,
n 2. *def.* 48. conſequently if FK ⊓⊔ √ FKq − CFq, *n* then
10. ſhall CK be a 2 bin. *o* wherefore H is a firſt 2 μ.
o 56. 10. laſtly, if FK ⊓⊔ √ FKq − CFq; *p* then CK
p 5. *def.* 48. ſhall be a fifth binom. *q* whence H ſhall contain
10. in power ῥν and μν. *Which was to be dem.*
q 59. 10.

PROP.

PROP. LXXIII.



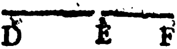
If two medial rectangles A, B, incommensurable to one another be compoſed together, the two remaining irrational lines are made, either a ſecond bimedial, or a line containing in power two medial rectangles.

As H containing in power A + B is one of the ſaid irrational lines. For upon CD propounded ῥ draw the rectangle CE = A, and FI = B, whence Hq = CI. Therefore becauſe CE and FI *a* are μα. *b* the latitudes CF, FK, ſhall be ῥ ⊔ CD, alſo becauſe CE *a* ⊔ FI, and CE. FI *c* :: CF. FK, *d* therefore CF ⊔ FK. *e* therefore CK is a 3. bin. namely, if CF ⊔ √ CFq − FKq, whence H = √ CI *f* ſhall be a ſecond 2 μ. But if CF ⊔ √ CFq − FKq, *g* then CK ſhall be a 6 binom. *h* and conſequently H contains in power 2 μα. *Which was to be dem.*

a *hyp.*
b 23. 10.
c 1. 6.
d 10. 10.
e 3. *def.* 48.
f 57. 10.
g 6. *def.* 48. 10.
h 60. 10.

Here begin the Senaries of lines irrational by Subtraction.

PROP. LXXIV.



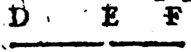
If from a rational line DF a rational line DE, commenſurable in power only to the whole DF, be taken away, the reſidue EF is irrational; and is called an Apotome or reſidual line.

For EFq *a* ⊔ DEq; *b* but DEq is ῥν; *c* therefore EF is ῥ. *Which was to be dem.*

In numbers, let there be DF, 2. DE, √3. then EF ſhall be 2 − √3.

a *lem.* 26. 10.
b *hyp.*
c 10. & 11. *def.* 10.

PROP. LXXV.



If from a medial line DF, a medial line DE commenſurable only in power to the whole DF, and comprehending with the whole DF a rational rectangle be taken away, the remainder EF is irrational, and is called a firſt reſidual line of a medial.

a *ſch.* 16. 10. For EFq *a* ⊤⊔ to the rectangle FDE. therefore ſeeing FDE *b* is ρ̇ν, *c* BF ſhall be ρ̇. *Which was to be dem.*

b *hyp.*

c 20. *and* 11. *def.* 10. In numbers, let DF be *v* √ 54, and DE *v* √ 24. therefore EF is *v* √ 54 — *v* √ 24.

PROP. LXXVI.

If from a medial line DF, a medial line DE be taken away being incommenſurable only in power to the whole DF, and comprehending together with the whole line DF a medial rectangle, the remainder EF is irrational, and is called a ſecond reſidual of a medial line.

a *hyp.* Becauſe DFq and DEq *a* are μ ⊤⊔; *b* there-
b 16. 10. fore ſhall DFq + DEq be ⊤⊔ DEq. *c* wherefore
c 24. 10. DFq + DEq is μν. alſo the rectangle FDE, *c*
d *cor.* 7. 1. and ſo 2 FDE, *a* is μν. therefore EFq (*d* DFq +
e 27. 10. DEq — 2 FDE) *e* is ρ̇ν. wherefore EF is ρ̇. *Which was to be dem.*

In numbers, let DF be *v* √ 18. and DE *v* √ 8. then EF *v* √ 18 — *v* √ 8.

PROP. LXXVII.

If from a right line AC be taken away a right line AB being incommenſurable in power to the whole BC, and making with the whole AC that which is compoſed of their ſquares rational, and the rectangle contained under them medial, the remainder BC is irrational, and is called a Minor line.

a *hyp.*
b *ſch.* 12. 10. For ACq + ABq *a* is ρν, but the rectangle ACB *a* is μν. *b* therefore 2 CAB ⊤⊔ ACq + ABq (2 *c*
c 7. 2.

CAB+BCq.) *d* therefore ACq+ABq ⊤⊥ BCq. *e* d 17. 10.
therefore BC is ῥ. *Which was to be dem.* e 11. *def.*
In numbers, let AC be √: 18 + √ 108 ; AB 10.
√: 18 − √ 108. then BC is √: 18 + √ 108 −
√: 18 − √ 108.

PROP. LXXVIII.

If from a right line DF be taken away a right line DE, being incommenſurable in power to the whole line DF, and with the whole DF making that which is compoſed of their ſquares medial, and the rectangle contained under the ſame lines rational, the line remaining EF is irrational, and is called a line making a whole ſpace medial with a rational ſpace.

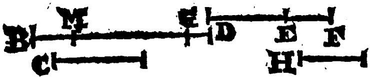
For 2 FDE *a* is ῥν. *b* and DFq+DEq is μν. *c* a *hyp. and*
therefore 2 FDE ⊤⊥ DFq+DEq *d* (2 FDE+EFq) *ſch.* 12.10.
e therefore EF is ῥ. *Which was to be dem.* b *hyp.*
In numbers, let DF be √: √ 216 + √ 72; DE c *ſch.* 12.
√: √ 216 − √ 72. therefore EF is √: √ 216 + 10.
√ 72 − √: √ 216 − √ 72. d 7. 2.
e *ſch.* 12.
10. & 11.
def. 10.

PROP. LXXIX.

If from a right line DF be taken away a right line DE incommenſurable in power to the whole DE, and which together with the whole makes that which is compoſed of their ſquares medial, and the rectangle contained under them, alſo medial and incommenſurable to that which is compoſed of their ſquares, the remainder is irrational, and is called a line making a whole ſpace medial with a medial ſpace.

For 2 FDE, and FDq+DEq *a* are μα; *b* there- a *hyp. and*
fore EFq (*c* DFq+DEq − 2 FDE) is ῥν. *d* and 24. 10.
ſo conſequently EF is ῥ. *Which was to be dem.* b 27. 10.
In numbers, let DF be √: √ 180 + √ 60. DE c *cor.* 7. 2.
√: √ 180 − √ 60. then EF ſhall be √: √ 180 d 11. *def.*
+ √ 60 − √: √ 180 − √ 10. 10.

Lemma.



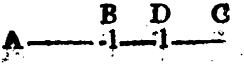
If there be the ſame exceſs between the firſt magnitude BG, and the ſecond C (MG) as is between the third magnitude DF and the fourth H (EF;) then alternately, the ſame exceſs ſhall be between the firſt magnitude BG and the third DF, as is between the ſecond C and the fourth H.

a *hyp.* For becauſe that *a* to the equals BM, DE, are added the unequals MG, EF, that is, C, H;
b 15. *ax.* 1. the exceſs of the wholes BG, DF, *b* ſhall be equal to the exceſs of the parts added C, H. *Which was to be dem.*

Coroll.

Hence, Four magnitudes Arithmetically proportional, are alternately alſo Arithmetically proportional.

PROP. LXXX.

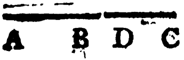


To an Apotome or reſidual line AB only one rational right line BC, being commenſurable in power only to the whole AB, is congruent, or can be joined.

a 22. 10. *ing commenſurable in power only to the whole AB,*
b 22. 10. *is congruent, or can be joined.*
c *cor.* 7. 2. If it be poſſible, let ſome other line BD be ad-
d *lem.* 79. ded to it; *a* then the rectangles ACB, ADB, *b*
10. and ſo conſequently the double of them are *μ.*
e *hyp.* and wherefore ſeeing ACq + BCq −: 2 ACB *c* = ABq *c*
27. 10. = ADq + DBq −: 2 ADB. therefore alternately
f *ſch.* 12. ACq + BCq −: ADq + DBq *d* = 2 ACB −: 2 ADB.
10. But ACq + BCq −: ADq + DBq *e* is *pr.* *f* therefore
g 27. 10. 2 ACB −: 2 ADB is *pr.* *Which is abſurd.*

PROP.

PROP. LXXXI.



To a firſt medial reſidual line AB only one medial right line BC, being commenſurable only in power to the whole, and comprehending with the whole line a rational rectangle, can be joined.

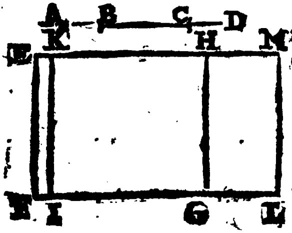
Conceive BD to be ſuch a line as may be joined to it; then becauſe ACq and BCq, as well as ADq and BDq *a* are μα ⊐⊏. *b* alſo ACq + BCq, a *hyp.*
and ADq+BDq ſhall be μα. *c* but the rectangles b 10. *and*
ACB, ADB, *d* and ſo 2 ACE and 2 ADB are ρα. 24. 10.
e therefore 2 ACB −: 2 ADB, *f* that is, ACq + c *hyp.*
BCq −: ADq+BDq is ρν, *g Which is abſurd.* d *ſch.* 12. 10.

e *ſch.* 27. 10.

PROP. LXXXII.

f 7. 2. *and lem.* 79. 10.

g 27. 10.

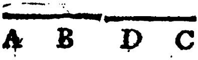


Unto a ſecond medial reſidual line AB only one medial right line BC, commenſurable only in power to the whole, and with it containing a medial rectangle, can be joined.

If it be poſſible, let ſome other line BD be added to it; and upon EF ρ, make the rectangle a 4. 2. *and*
EG = ACq + BCq; as alſo the rectangle EL = 3. *ax.* 1.
ADq + BDq. likewiſe EI = ABq. Now 2 ACB b *hyp.*
+ ABq = ACq + BCq = EG. therefore ſeeing EI c 24. 10.
= ABq, *a* alſo GK ſhall be = 2 ACB. moreover d 23. 10.
ACq and BCq *b* are μα ⊐⊏. *c* therefore EG (ACq e *hyp.*
+ CBq) is μν. *d* therefore the breadth EH is ρ f 24. 10.
⊐⊏ EF. *e* Further, the rectangle ACB *f* and ſo g *lem.* 26. 10.
2 ACB (KG) is μν. *d* therefore KH is alſo B, ρ
EF. laſtly, becauſe ACq + BCq (EG) *g* ⊐⊏ 2 ACB
(KG) and EG. KG *h* :: EH. KH. *k* therefore h 1. 6.
EH ⊐⊏ HK. *l* therefore EK is a reſidual line, k 10. 10.
whereto HK is congruent. by the ſame reaſon l 74. 10.

alſo ſhall KM be congruent to the ſaid EK. *which is repugnant to the* 80. *Prop. of this Book.*

PROP. LXXXIII.

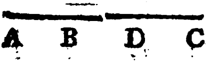


To a Minor line AB only one right line BC can be joined being incommenſurable in power to the whole; and making together with the whole line that which is compoſed of their ſquares rational, and the rectangle which is contained under them medial.

Conceive any other BD to be congruent to it; therefore whereas ACq + BCq, and ADq + DBq *a* are ῥα, their exceſs (2 *b* ACB −: 2 ADB) *c* is ῥν, *Which is abſurd*; becauſe ACB and ADB are μα by the Hyp.

a *hyp.*
b *lem.* 97. 10.
c *ſch.* 27. 10.
d 27. 10.

PROP. LXXXIV.



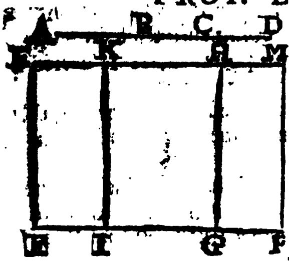
Unto a line (AB) making with a rational ſpace a whole ſpace medial only one right line BC can be joined, being incommenſurable in power to the whole, and making together with the whole that which is compoſed of their ſquares medial, and the rectangle which is contained under them rational.

Suppoſe ſome other BD to be congruent alſo to it; *a* then the rectangles ACB, ADB, *b* and ſo 2 ACB and 2 ADB are ῥα. therefore 2 ACB −: 2 ADB, *c* that is, ACq+BCq −: ADq+BDq *d* is ῥν. *Which is abſurd.* ſince ACq + BCq, and ADq + BDq are μα by the Hyp.

a *hyp.*
b *ſch.* 12. 10.
c *lem.* 79. 10.
d *ſch.* 27. 10.

PROP.

PROP. LXXXV.



To a line AB, which with a medial ſpace makes a whole ſpace medial, can be joined only one right line BC, incommenſurable in power to the whole, and making with the whole both that which is compoſed of their ſquares medial, and the rectangle which is contained under them medial and incommenſurable to that which is compoſed of their ſquares.

Thoſe things being ſuppoſed which are done and ſhewn in the 82. Prop. of this Book; it is clear that EH and KH are ρ ⊔ EF. Beſides, being that ACq+CBq, that is, the rectangle EG, *a* is ⊔ ACB. *b* and ſo EG ⊔ 2 ACB (KG;) and EG.KG *c* :: EH.KH; ſhall EH be ⊔ KH. therefore EK is a reſidual line, and the line congruent to it is KH. In like manner may KM be ſhewn to be congruent to the ſaid reſidual EK, againſt the 80. Prop. of this Book.

a *hyp.*
b 14. 10.
c 1. 6.

Third Definitions.

A Rational line and a reſidual being propounded, if the whole be more in power than the line joined to the reſidual, by the ſquare of a right line commenſurable unto it in length; then

I. If the whole be commenſurable in length to the rational line propounded, it is called a firſt reſidual line.

II. But if the line adjoined be commenſurable in length to the rational line propounded, it is called a ſecond reſidual line.

III. If neither the whole nor the line adjoined be commenſurable in length to the rational line propounded, it is called a third reſidual line.

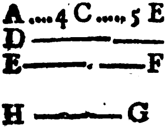
Moreover, if the whole be in more power than the line adjoined by the ſquare of a right line incommenſurable to it in length, then

IV. If the whole be commenſurable in length to the rational line propounded, it is called a fourth reſidual line.

V. But if the line adjoined be commenſurable in length to the rational line propounded, it is a fifth reſidual,

VI. If neither the whole nor the line adjoined be commenſurable in length to the rational line propounded, it is termed a ſixth reſidual line.

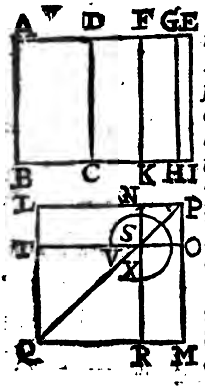
PROP. LXXXVI, 87, 88, 89, 90, 91.



To find out a firſt, ſecond, third, fourth, fifth, and ſixth reſidual line.

Reſidual lines are found out by ſubducting the leſs names or parts of binomials from the greater. *ex. gr.* Let $6 + \sqrt{20}$ be a firſt binom. then ſhall $6 - \sqrt{20}$ be a firſt reſidual. So that it is not neceſſary to repeat more concerning the finding of them out.

Lemma.



Let AC be a rectangle contained under the right lines AB, AD. Let AD be drawn forth to E, and DE equally divided in F. and let the rectangle AGE be = FEq, and the rectangles AI, DK, FH, be finiſhed. Then let the ſquare LM = AH be made, and the ſquare NO = GI; and the lines NSR, OST, produced.

I ſay 1. the rectangle AI = LM + NO = TOq + SOq. which appears by the conſtr.

2. *The rectangle DK = LG.* For becauſe the
rectangle AGE *a* = FEq. *b* thence are AG, FE, a *conſtr.*
GE, ∺ *c* and ſo AH, FI, GI ∺, *d* that is, LM, FI, b 17. 6.
NO ∺; but LM, LO, NO *d* are ∺; therefore c 1. 6.
FI = *e* LO *f* = DK = *g* NM. d *ſch.* 32. 6.

3. *Hence, AC = AI DK − FI = LM + NO − LO* e 9. 5.
− NM = TR. f 36. 1.

4. *h It is manifeſt that DF, FE, DE, are* ⊓⊔. g 43. 1.

5. *If AE* ⊓⊦ *DE, and AE* ⊓⊔ √ *AEq − DEq,* h 16. 10.
k then ſhall AG, GE, AE be ⊓⊔. k 18. *and*

6. *Alſo, becauſe AE l* ⊓⊔ *DE, m thence ſhall* 10. 10.
AE, FE be ⊓⊔. *n and ſo AI, FI, that is, LM +* l *hyp.*
NO and LO are ⊓⊔. m 13. 10.

7. *Becauſe AG* * ⊓⊔ *GE, n ſhall AH, GI, that* n 1. 6. *and*
is, LM, NO be ⊓⊔. 10. 10.

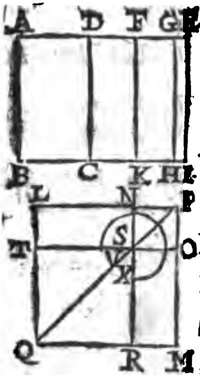
8. *But becauſe AE l* ⊓⊔ *DE, o therefore ſhall* * *before*
FE, GE be ⊓⊔, *n and ſo the rectangle FI* ⊓⊔ *GI,* o 14. 10.
that is, LO ⊓⊔ *NO. wherefore ſeeing LO. NO p* ∷ p 2. 6.
TS. SO. q therefore ſhall TS, SO be ⊓⊔. q 10. 10.

9. *If AE be put* ⊓⊔ √ *AEq − DEq, r then* r 19. 10. &
ſhall AG, GE, AE be ⊓⊔. 17. 10.

10. *ſ Wherefore the rectangles AH, GI, that is,* ſ 1. 6. *and*
TOq, SOq ſhall be ⊓⊔. 10. 10.

PROP.

PROP. XCII.



If a ſpace AC be contained under a rational line AB, and a firſt reſidual line AD (AE − DE) the right line TS, which contains the ſpace AC in power, is a reſidual line.

Uſe the foregoing *Lemma* for a preparatory to the demonſtration of this Prop. Therefore TS = √ AC. Alſo AG, GE, AE, are ⊏⊐; therefore ſince AE ⊏⊐ *a* AB

a *hyp.* $\overset{s}{\rho}$, *b* alſo AG and GE ſhall
b 12. 10. be ⊏⊐ AB. *c* therefore the
c 20. 10. rectangles AH and GI, that
d *lem.* 91. is, TOq and SOq are $\overset{s}{\rho}\alpha$. *d* Likewiſe TO, SO,
10. are $\overset{s}{\rho}$ ⊏̸⊐. *e* and conſequently TS is a reſidual
e 74. 10. line. *Which was to be dem.*

PROP. XCIII.

See the preceding Scheme.

If a ſpace AC be contained under a rational line AB and a ſecond reſidual AD (AE − DE) the right line TS, containing the ſpace AC in power, is a firſt medial reſidual line.

Again, by the foregoing *Lemma*, AG, GE,
a *hyp.* AE are ⊏⊐. therefore *a* ſince AE is $\overset{s}{\rho}$ ⊏⊐ AB,
b 13. 10. *b* alſo AG, GE, ſhall be $\overset{s}{\rho}$ ⊏⊐ AB. *c* therefore
c 22. 10. the rectangles AH, GI, that is, TOq, SOq are μα
d *lem.* 74. *d* likewiſe TO ⊏̸⊐ SO. Laſtly, becauſe DE *e*
10. ⊏⊐ AB $\overset{s}{\rho}$, *f* the right angle DI, and the half
e *hyp.* thereof DK or LO, that is, TOS ſhall be $\overset{s}{\rho}\nu$. *g*
f 20. 10. from whence it follows that TS (√ AC) is a
g 75. 10. firſt medial reſidual. *Which was to be dem.*

PROP. XCIV.

See Scheme 92.

If a ſpace AC be contained under a rational line AB and a third reſidual AD (AE — DE) the right line TS containing in power the ſpace AC is a ſecond medial reſidual line.

As in the former, TO and SO are μ. Therefore becauſe DE *a* is ῥ ⊤⊤ AB. *b* the rectangle a *hyp.*
DI, *c* and ſo DK, or TOS, ſhall be μν. *d* there- b 21. 10.
fore TS = √ AC is a ſecond medial reſidual. c 24. 10.
Which was to be dem. d 76. 10.

PROP. XCV.

See Scheme 92.

If a ſpace AC be contained under a rational line AB and a fourth reſidual AD (AE — DE) the right line TS containing the ſpace AC in power, is a Minor line.

As before, TO *a* ⊤⊏ SO. Therefore becauſe a *lem.* 91.
AE *b* is ῥ ⊐⊏ AB, *c* ſhall AI (TOq + SOq) be 10.
ῥν. but, as before, the rectangle TOS is μν. *d* b *hyp.*
therefore TS = √ AC is a Minor line. *Which* c 20. 10.
was to be dem. d 77. 10.

PROP. XCVI.

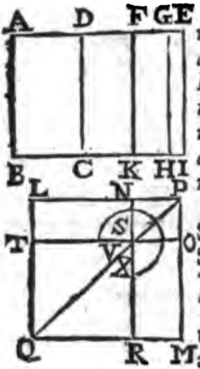
See Scheme 92.

If a ſpace AC be contained under a rational line AB and a fifth reſidual AD (AE — DE) the right line TS containing in power the ſpace AC, is a line which makes with a rational ſpace the whole ſpace medial.

For again TO ⊤⊏ SO. therefore ſince AE *a* is a *hyp.*
ῥ ⊐⊏ AB. *b* alſo AI, that is, TOq+SOq, ſhall b 22. 10.
be μν. But, as in the 93. the rectangle TOS is ῥν. c 78. 10.
c whence TS = √ AC is a line which with ῥν
makes a whole μν. *Which was to be dem.*

PROP.

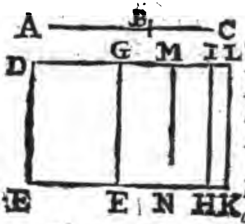
PROP. XCVII.



If a ſpace AC be contained under a rational line AB, and a ſixth reſidual AD (AE — DE) the right line TS containing in power the ſpace AC, is a line making with a medial rectangle, a whole ſpace medial.

As often above, TO ⊐⊏ SO. alſo, as in 96. TOq + SOq is μν. but the rectangle TOS is ρν, as in 94. a lem. 91. 10. *a* Laſtly, TOq + SOq ⊐⊏ TOS. b 79. 10. *b* therefore TS = √ AC is a line which with μν makes a whole μν. *Which was to be dem.*

Lemma.



* cor. 16.6. *Upon a right line DE * apply the rectangles DF = ABq, and DH = ACq, and IK = BCq. and let GL be biſected in M, and the line MN drawn parallel to GF.*

Then 1. *The rectangle DK is = ACq + BCq. as the conſtruction manifeſts.*

a conſtr. b 7. 2. c 3. ax. 1. d 7. ax. 1. 2. *The rectangle* ACB = GN or MK. For DK *a* = ACq + BCq *b* = 2 ACB + ABq. but ABq *a* = DF. therefore GK *c* = 2 ACB. and *d* conſequently GN or MK = ACB.

e 1. 6. f 17. 6. 3. *The rectangle* DIL = MLq. For becauſe ACq. ACB *e* :: ACB. BCq, that is, DH, MK :: MK. IK. *e* thence is DI.ML :: ML.IL. *f* therefore DIL = MLq.

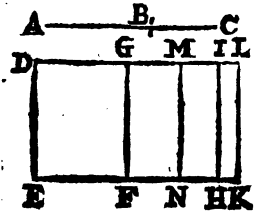
4. *If AC be taken* ⊐⊏̦ *BC, then DK ſhall be* ⊐⊏ *ACq*. For ACq + BCq (DK) *g* ⊐⊏ ACq. g 16. 10.

5. *Likewiſe* DL ⊐⊏ √DLq−GLq. For becauſe DH (ACq) ⊐⊏ IK (BCq) *h* thence ſhall DI be ⊐⊏ IL. *k* therefore √DLq−GLq ⊐⊏ DL. h 10. 10. k 18. 10.

6. *Alſo* DL ⊐⊏ GL. For ACq + BCq ⊐⊏ *l* 2 ACB. that is, DK ⊐⊏ GK. *m* therefore DL ⊐⊏ GL. l lem. 26. 10. m 10. 10.

7. *But if AC be taken* ⊐⊏̦ *BC, n then DL ſhall be* ⊐⊏ *√DLq − GLq.* n 19. 10.

PROP. XCVIII.



The ſquare of a reſidual line AB (AC−BC) applied to a rational line DE, makes the breadth DG a firſt reſidual line.

Do as is injoined in the *Lemma* next preceding. Then becauſe AC, BC, *a* are ρ̇ ⊐⊏̦ : *b* alſo DK (ACq+BCq) ſhall be ⊐⊏ ACq. *c* therefore DK is ρ̇ν. *d* wherefore DL is ρ̇ ⊐⊏ DE. *e* Likewiſe the rectangle GK (2 ACB) is μν. *f* therefore GL is ρ̇ ⊐⊏ DE. *g* and conſequently DL ⊐⊏ GL. *h* But DLq ⊐⊏ GLq. *k* therefore DG is a reſidual, *l* and that of the firſt order (becauſe *m* AC ⊐⊏̦ BC, and therefore DL ⊐⊏ √DLq−GLq.) *Which was to be dem.*

a hyp.
b lem. 97. 10.
c ſch. 12. 10.
d 21. 10.
e 22. and 24. 10.
f 23. 10.
g 13. 10.
h ſch. 12. 10.
k 74. 10.
l 1. def. 85. 10.
m lem. 97. 10.

PROP.

PROP. XCIX.

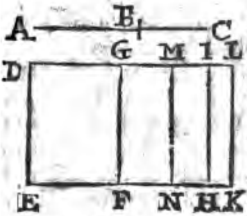
See the following Scheme.

The ſquare of a firſt medial reſidual line AB (AC — BC) applied to a rational line DE, makes the breadth DG a ſecond reſidual line.

Suppoſing the foregoing *Lemma*; becauſe AC
a *hyp.* and BC *a* are μ ⊓, *b* thence ſhall DK (ACq +
b *lem.* 97. BCq) be ⊔ ACq. *c* wherefore DK is μν. *d*
10. therefore DL is ρ ⊔ DE. *e* alſo GK (2 ACB)
c 24. 10. is μν *f* therefore GL is ρ ⊔ DE, *g* wherefore
d 23. 10. DL ⊔ GL. *h* But DLq ⊔ GLq. *k* therefore
e *hyp.* and DG is a reſidual line: and becauſe DL is ⊔ √
ſch. 12. 10. DLq — GLq. *m* therefore ſhall DG be a ſecond
f 21. 19. reſidual. *Which was to be dem.*
g 13. 10.
h *ſch.* 12.
10.

PROP. C.

k 74. 10.
l *lem.* 97.
10.
m 2. *def.*
85. 10.



The ſquare of a ſecond medial reſidual line AB (AC — BC) applied to a rational line DE makes the breadth DG a third reſidual line.

a 23. 10. Again DK is μν. *a*
b *lem.* 26. wherefore DL is ρ ⊔
10. DE. alſo DGK is μν. *a*
c 1. 6. and whence GL is ρ ⊔ DE. *b* likewiſe DK ⊔
10. 10. GK. *c* wherefore DL ⊔ GL. *d* but DLq ⊔
d *ſch.* 12. GLq. *e* therefore DG is a reſidual line, and that
10. of *f* the third order, *g* becauſe DL ⊔ √ DLq
e 74. 10. — GLq. *Which was to be dem.*
f 3. *def.* 85.
10.
g *lem.* 97.
10.

PROP. CI.

See the foregoing Scheme.

The ſquare of a Minor line AB (AC—BC) applied to *to a rational line DE, makes the breadth DG a fourth reſidual.*

As before, ACq + BCq, that is, DK is *ρ̇ν. a* — a 21. 10.
therefore DL is ρ̇ ⊓⊔ DE. but the rectangle ACB,* — * *hyp.*
and ſo GK (2 ACB) * is *μν. b* wherefore GL is — b 23. 10.
ρ̇ ⊓⊔ DE, *c* therefore DL ⊓⊔ GL. *b* but — c 13. 10.
DLq ⊓⊔ GLq. and becauſe * ACq ⊓⊔ BCq, *e* — d *ſch.* 12.
thence ſhall DL be ⊓⊔ √ DLq − GLq. *f* there- — 10.
fore DG has the conditions required to a fourth — e *lem.* 97.
reſidual. *Which was to be dem.* — 10.
f 4.*def.*85.
10.

PROP. CII.

See Scheme 100.

The ſquare of a line AB (AC − BC) which makes with a rational ſpace the whole ſpace medial, applied to a rational line DE, makes the breadth DG a fifth reſidual line.

For, as above, DK is *μν. a* wherefore DL is ρ̇ — a 23. 10.
⊓⊔ DE. alſo GK is *ρ̇ν. b* whence GL is ρ̇ ⊓⊔ — b 21. 10.
DE. *c* therefore DL ⊓⊔ GL. *d* but DLq ⊓⊔ GLq. — c 13. 10.
Moreover DL *e* ⊓⊔ √ DLq − GLq. wherefore — d *ſch.* 12.
DG *f* is a fifth reſidual. *Which was to be dem.* — 10.
e *lem.* 97.
10.
f 5.*def.*85.
10.

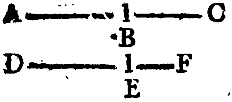
PROP. CIII.

See the laſt Scheme.

The ſquare of a line AB (AC−BC) making with a medial ſpace the whole ſpace medial, applied to a rational line DE, makes the breadth DG a ſixth reſidual line.

As above DK and GK are *μα*; *a* wherefore — a 23. 10.
DL and GL are ρ̇ ⊓⊔ DE. alſo DK *b* ⊓⊔ GK. — b *hyp. and*
c whence DL ⊓⊔ GL. *d* therefore DG is a re- — *lem.* 97. 10.
ſidual. *b* And whereas ACq ⊓⊔ BCq. and ſo DL — c 10. 10.
⊓⊔ √ DLq − GLq, *e* therefore DG ſhall be a — d 74. 10.
ſixth reſidual. *Which was to be dem.* — e 6.*def.*85.
10.

PROP. CIV.



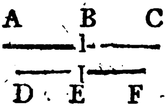
A right line DE commenſurable in length to a reſidual AB (AC — BC) is it ſelf alſo a reſidual, and of the ſame order.

Lemma.

Let AB. DE :: AC. DF. *and* AB ⊓⊔ DE.
I ſay AC + BC ⊓⊔ DF + EF. For AC. BC
a :: DF. EF, therefore by addition AC + BC.
BC :: DF + EF. FE, therefore by inverſion AC
a *lem.* 66. + BC. DF + EF :: BC. EF. *a* but BC ⊓⊔ EF.
10. *b* therefore AC + BC ⊓⊔ DF + EF. *Which*
b 10. 10. *was to be dem.*

a 12. 6. *a* Make AB. DE :: AC. DF. *b* therefore AC +
b *lem.* 103. BC ⊓⊔ DF + EF. therefore ſeeing AC + BC *c*
10. is a binomial, *d* DF + EF ſhall be a binomial
c *hyp.* too, and of the ſame order. *e* wherefore DF —
d 67. 10. EF is a reſidual of the ſame order with AC —
e *by def.* BC. *Which was to be dem.*
85. 10.

PROP. CV.

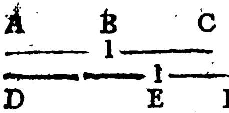


A right line DE commenſurable to a medial reſidual line AB (AC — BC) is it ſelf a medial reſidual, and of the ſame order.

a 12. 6. Again *a* make AB. DE :: AC. DF. *b* whence
b *lem.* 103. AC + BC ⊓⊔ DF + EF. *c* therefore DF + EF
10. is a bimedial of the ſame order with AC + BC,
c 68. 10. *d* and conſequently DF — EF ſhall be a medial
d 75. *and* reſidual of the ſame order with AC + BC. *Which*
76. 10. *was to be demonſtrated.*

PROP.

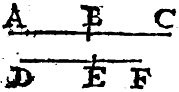
PROP. CVI.



A right line DE commenſurable to a Minor line AB (AC — BC) is it ſelf alſo a Minor line.

Make AB. DE :: AC. DF. *a* then is AC + BC a *lem.*103.
⊔ DF + EF. But AC + BC *b* is a Major line ; 10.
c therefore DF + EF is alſo a Major line ; *d* and b *hyp.*
conſequently DF — EF is a Minor line. *Which* c 69. 10.
was to be dem. d 77. 10.

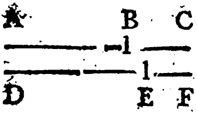
PROP. CVII.



A right line DE commenſurable to a line AB (AC — BC) which makes with a rational ſpace the whole ſpace medial, is it ſelf alſo a line making with a rational ſpace the whole ſpace medial.

For, accordingly as in the former, we may ſhew DF + EF to contain in power ῥν and μν. *a* whence DF — EF is a line making, &c. a 78. 10.

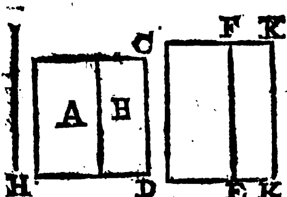
PROP. CVIII.



A right line DE commenſurable to a line AB (AC — BC) which with a medial ſpace makes the whole ſpace medial, is it ſelf a line making with a medial ſpace the whole ſpace medial.

For according to the preceding DF + EF ſhall contain in power 2 μμ. *a* therefore DF — EF ſhall be, as in the Prop. a 79. 10.

PROP. CIX.



A medial rectangle B being taken from a rational rectangle A+B, the right line H which contains in power the ſpace remaining A, is one of thoſe two irrational lines, viz. either a reſidual line, or a Minor line.

Upon CD $\dot{\rho}$ make the rectangles $CI = A + B$.
a 5. *ax.* 1. and $FI = B$. whence $CE \overset{a}{=} A = Hq$. wherefore
b *hyp. and* becauſe CI *b* is $\dot{\rho}\nu$. *c* therefore CK is $\dot{\rho}$ ⊓⊔ CD.
conſtr. but being FI *b* is $\mu\nu$, *d* ſhall FK be $\dot{\rho}$ ⊓⊔ CD. *e*
c 21. 10. whence CK ⊓⊔ FK. *f* therefore CF is a reſidual
d 23. 10. line. Wherefore if CK be ⊓⊔ $\sqrt{CKq - FKq}$,
e 13. 10. *g* then CF ſhall be a firſt reſidual, *h* therefore $\sqrt{}$
f 74. 10. CE (H) is a reſidual line. But if CK ⊓⊔ $\sqrt{}$
g 1. *def.* 85. $CKq - FKq$. *k* then CF ſhall be a fifth reſi-
10. dual; and conſequently H ($\sqrt{CE}$) *l* ſhall be a
h 92. 10. Minor line. *Which was to be dem.*
k 4. *def.* 85.
10.

PROP. CX.

l 95. 10. *See the prec. Scheme.*

A rational rectangle B being taken away from a medial rectangle A+B, other two irrational lines are made, namely either a firſt medial reſidual line,
a 3. *ax.* 1. *or a line making with a rational ſpace the whole*
b *hyp. and ſpace medial.*
conſtr. Upon CD the propounded $\dot{\rho}$ make the rectan-
c 23. 10. gles $CI = A + B$, and $FI = B$. *a* whence $CE = A$
d 21. 10. $= Hq$. Therefore becauſe CI *b* is $\mu\nu$, *c* ſhall CK
e 13. 10. be $\dot{\rho}$ ⊓⊔ CD. but becauſe FI *b* is $\dot{\rho}\nu$, *d* thence FK
f 74. 10. $\dot{\rho}$ ⊓⊔ CD. *e* whence CK ⊓⊔ FK. *f* therefore CF
g 2. *def.* 85. is a reſidual, *g* and that a ſecond. If CK ⊓⊔
10. $\sqrt{CKq - FKq}$, *h* then H ($\sqrt{CE}$) is a firſt me-
h 93. 10. dial reſidual. But if CK ⊓⊔ $\sqrt{CKq - FKq}$. *k*
k 5. *def.* 85. then ſhall CF be a fifth reſidual; and *l* conſe-
10. quently H ($\sqrt{CE}$) ſhall be a line making $\mu\nu$
l 96. 10. with $\dot{\rho}\nu$. *Which was to be dem.*

PROP.

PROP. CXI.

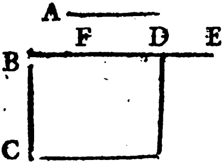
See the ſame Scheme.

A medial ſpace B being taken away from a medial ſpace A + B, which is incommenſurable to the whole A + B, the other two irrational lines are made, viz. *either a ſecond medial reſidual line, or a line making with a medial ſpace the whole ſpace medial.*

Upon CD ρ make the rectangles CI = A + B, and FI = B. *a* wherefore CE = A = Hq. Becauſe therefore CI is μν, *b* thence CK is ρ ˥⊔ CD, and in like manner FK ρ ˥⊔ CD. Likewiſe becauſe CI *c* ˥⊔ FI, *d* therefore CK ˥⊔ FK. *e* wherefore CF is a reſidual, *f* namely a third. If CK ˥⊔ √ CKq − FKq, *g* whence H (√ CE ſhall be a ſecond medial reſidual, but if CK ˥⊔ √ CKq − FKq *h* then ſhall CF be a ſixth reſidual. *k* wherefore A ſhall be a line making μν with μ. *Which was to be dem.*

a 3. *ax.* 1.
b 23. 10.
c *hyp.*
d 10. 10.
e 74. 10.
f 3.*def.*85. 10.
g 94. 10.
h 6.*def.*85. 10.
k 97. 10.

PROP. CXII.



A reſidual line A is not the ſame with a bimedial line.

Upon BC propounded ρ make the rectangle CD = Aq. Therefore ſeeing A is a reſidual, *a* BD ſhall be a firſt reſidual, to which let DE be the line congruent, or that may be adjoined. *b* wherefore BE, DE, are ρ ˥⊤. *c* and BE ˥⊔ BC. If you conceive A to be a binomial, then BD is a firſt bin. whoſe names let be BF, FD; and let BF be ⊏ FD. *d* therefore BF, FD are ρ ˥⊤; and BF *e* ˥⊔ BC. therefore ſince BC ˥⊔ BE *f* ſhall BE be ˥⊔ BF. *g* and thence BE ˥⊔ FE. *h* therefore FE is ρ. Likewiſe becauſe BE ˥⊔ DE, *k* ſhall FE be ˥⊔ DE. *l* wherefore FD is a reſidual, and ſo FD is ρ. but it was ſhewn ρ, which are repugnant. Therefore A is falſly conceived to be a binomial. *Which was to be dem.*

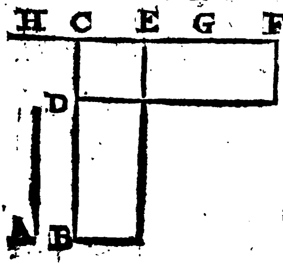
a 98. 10.
b 74. 10.
c 1.*def.*85. 10.
d 37. 10.
e 1.*def.*48. 10.
f 12. 10.
g *cor.*16.10
h *ſch.* 12. 10.
k 14. 10.
l 74. 10.

The names of the 13 irrational lines differing one from another.

1. A Medial line.
2. A binomial line; of which there are ſix ſpecies.
3. A firſt bimedial line.
4. A ſecond bimedial.
5. A Major line.
6. A line containing in power a rational ſuperficies, and a medial ſuperficies.
7. A line containing in power two medial ſuperficies.
8. A reſidual line; of which there are alſo 6 kinds.
9. A firſt medial reſidual line.
10. A ſecond medial reſidual line.
11. A Minor line.
12. A line making with a rational ſuperficies the whole ſuperficies medial.
13. A line making with a medial ſuperficies the whole ſuperficies medial.

Being the differences of breadths do argue differences of right lines, whoſe ſquares are applied to ſome rational line, and it is demonſtrated in the prec. Propoſitions that the breadths which ariſe from applying of the ſquares of theſe 13 lines do differ one from another, it evidently follows that theſe 13 lines do alſo differ one from another.

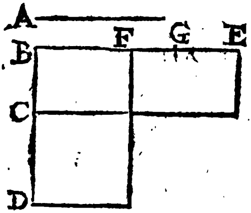
PROP. CXIII.



The ſquare of a rational line A applied to a binomial BC (BD + DC) makes the breadth EC a reſidual line, whoſe names EH, CH, are commenſurable to the names BD, DO, of the binomial line, and in *in the ſame proportion (EH. BD :: CH. DC.) and moreover the reſidual line EC which is made, is of the ſame order with BC the binomial.*

Upon DC the leſs name *a* make the rectangle a *cor.*16.6.
DF = Aq = BE. whence BC. CD *b* :: FC. CE. b 14. 6.
therefore by diviſion, BD. DC :: FE. EC. And
whereas BD *c* ⊏ DC, *d* thence FE ſhall be ⊏ c *hyp.*
EC. Take EG = EC, and make FG. GE :: EC. d 14. 5.
CH. Then EH, and CH ſhall be the names of
the reſidual EC, whereunto all is agreeable that
is propounded in the Theorem. For being that
by addition FE. GE (EC) :: EH. CH. therefore e 12. 5.
FH, EH *e* :: EH. CH *f* :: FE. EC *f* :: BD. BC, f *before.*
wherefore ſince BD *g* ᅲ DC. *h* thence ſhall EH g *hyp.*
be ᅲ CH. *h* and FHq ㅍ EHq. Therefore be- h 10. 10.
cauſe FHq. EHq :: FH. CH. *k* ſhall FH be ㅍ k *cor.*20.6.
CH, *l* and ſo FC ㅍ CH. Moreover CD *g* is ῥ, l 16. 10.
and DF (Aq) *g* is ῥv. *m* therefore FC is ῥ, ㅍ m 21. 10.
CD. whence alſo CH is ῥ ㅍ CD. *n* therefore n *ſch.* 12.
EH, CH are ῥ and ᅲ. as before, *o* therefore EC 10.
is a reſidual line, to which CH may be joined. o 74. 10.
Furthermore EH. CH *f* :: BD. DC. and ſo by
inverſion EH. BD :: CH. DC. whence becauſe
CH *f* ㅍ DC, *p* ſhall EH be ㅍ BD. But ſup- p 10. 10.
poſe BD ㅍ √ BDq − DCq. *q* then ſhall EH be q 15. 10.
ㅍ √ EHq − CHq. Alſo if BD ㅍ ῥ propound- r 12. 10.
ed, then ſhall EH be ㅍ to the ſame ῥ. *ſ* that ſ 1. *def.*48.
is, if BC be a firſt binomial, *t* EC ſhall be a firſt 10.
reſidual. In like manner, if DC be to the ㅍ t 1. *def.*85.
propounded ῥ. *r* then is CH ㅍ to the ſame ῥ. *u* 10.
that is, if BC be a ſecond binomial, *x* EC ſhall u 2. *def.*48.
be a ſecond reſidual; and if this be a third binom. 10.
then that ſhall be a third reſidual, *&c.* But if x 2. *def.*85.
BD be ㅍ √ BDq − DCq, *y* then ſhall EH be 10.
ㅍ √ EHq − CHq. therefore if BC be a 4th, y 15. 10.
5th, or 6th binomial, EG ſhall be likewiſe a 4th,
5th or 6th reſidual. *Which was to be dem.*

PROP. CXIV.

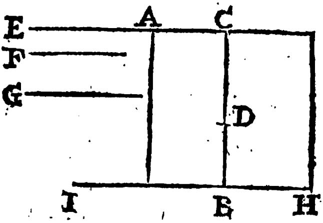


The ſquare of a rational line A applied to a reſidual line B C (BD – CD) makes the breadth BE a binomial; whoſe names BE, GE are commenſurable to the names BD, BC of the reſidual line BC, and in the ſame proportion, and moreover, the binomial line which is made (BE) is of the ſame order with the reſidual line (BC.)

a cor.16.6. *a* Make the rectangle DF = Aq. and BF. FE
b 12.6. *b* :: EG. GF. whence for that DF=Aq=CE, *c*
c 14.6. therefore BD. BC :: BE. BF, therefore by con-
d 19.5. verſion of proportion BD. CD :: BE. FE :: EG.
e *hyp.* GF :: *d* BG. EG. but BD *e* ⊤⊣ CD. *f* therefore
f 10.10. BG ⊤⊣ GE. therefore becauſe BGq. GEq *g* ::
g *cor.*20.6. BG. GF. *h* ſhall BG be ⊤⊔ GF. *k* and ſo BG
h 10.10. ⊤⊔ BF. moreover BD *e* is ρ, and the rectangle
k *cor.* 16. DF (Aq) *e* is ρν. *l* therefore BF is ρ ⊤⊔ BD. *m*
10. therefore alſo BG is ρ ⊤⊔ BD. *n* therefore BG,
l 21.10. GE are ρ ⊤⊣. *o* wherefore BE is a binomial.
m 12.10. Laſtly, becauſe BD. CD :: BG. GE. and inverſe-
n *ſch.* 12. ly BD. BG :: CD. GE. and BD ⊤⊔ BG. *p* thence
10. ſhall CD be ⊤⊔ GE. therefore if CB be a firſt
o 37.10. reſidual, BE ſhall be a firſt binomial, *&c.* as in
p 10.10. the prec. therefore, *&c.*

PROP.

PROP. CXV.



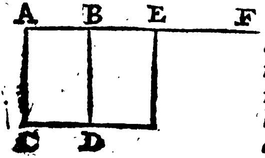
If a ſpace AB be contained under a reſidual line AC (CE — AE) and a binomial CB, whoſe names CD, DB are commenſurable to the names CE, AE, of the reſidual line, and in the ſame proportion (CE. AE :: CD. DB) then the right line F which contains in power that ſpace AB, is irrational.

Let G be ρ̇. and make the rectangle CH =
Gq; *a* then ſhall BH (HI — IB) be a reſidual line, a 113. 10.
and HI *a* ⊤⊔ CD *b* ⊤⊔ CE. *a* and BI ⊤⊔ DB a b *hyp.*
and HI.BI :: CD.DB *b* :: CE.EA. therefore by in-
verſion HI. CE :: BI. EA. *c* therefore BH. AC :: c 19. 5.
HI. CE :: BI.EA. *c* therefore BH.AC :: HI. CE :: d 12. 10.
BI. EA. wherefore ſince *d* HI ⊤⊔ CE, *e* thence e 10. 10.
BH ⊤⊔ AC. *f* therefore the rectangle HC ⊤⊔ BA. f 1. 6. *and*
But HC (Gq) *b* is ρ̇ν, *g* therefore BA (Fq) is ρ̇ν : g *ſch.* 12.
and conſequently F is ρ̇. *Which was to be dem.* 10.

Coroll.

Hereby it appears that a rational ſuperficies may be contained under two irrational right lines.

PROP. CXVI.

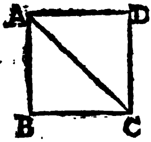


Of a medial line AB are produced infinite irrational lines BE, EF, &c. whereof none is of the ſame kind with any of the precedent.

Let AC be propounded ρ̇. and AD a rectangle

contained under AC, AB. *a* therefore AD is $\overset{s}{\rho}r$.
a *lem.* 38. 10. Take BE = √ AD. *b* then BE is $\dot{\rho}$, and the ſame with none of the former. For no ſquare of
b 11. 10. any of the former being applied to $\dot{\rho}$, makes the breadth medial. Let the rectangle DE be finiſhed, *a* then DE ſhall be $\overset{s}{\rho}r$. and *b* conſequently EF (√ DE) ſhall be $\dot{\rho}$, and not the ſame with any of the former, for no ſquare of the former being applied to $\overset{s}{\rho}$, makes the breadth BE. therefore, &c.

PROP. CXVII.



Let it be required to ſhew that in ſquare figures BD. the diameter AC is incommenſurable in length to the ſide AB.

a 47. 1. b cor. 24. 8. c 9. 10. For ACq. ABq *a* :: 2. 1 *b* :: not Q. Q. *c* therefore AC $\sqcup$ AB. *Which was to be dem.* This Theorem was of great note with the ancient Philoſophers; ſo that he that underſtood it not was eſteemed by *Plato* undeſerving the name of a Man, but rather to be reckoned among Brutes.

The End of the tenth Book.

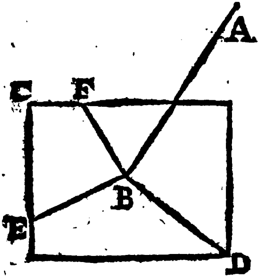
THE

THE ELEVENTH BOOK OF EUCLIDE'S ELEMENTS.

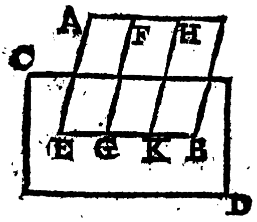
Definitions.

A Solid is that which hath length, breadth, and thickneſs.

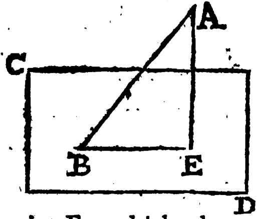
II. The term, or extreme of a ſolid is a Superficies.



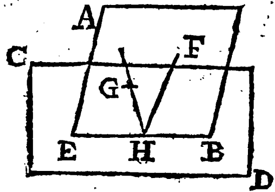
III. A right line AB is perpendicular to a Plane CD, when it makes right angles ABD, ABE, ABF, with all the right lines BD, BE, BF, that touch it, and are drawn in the ſaid Plane.



IV. A Plane AB, is perpendicular to a Plane CD, when the right lines FG, HK, drawn in one Plane AB to the line of common ſection of the two Planes EB, and making right angles therewith, do alſo make right angles with the other Plane CD.



V. The inclination of a right line AB to a Plane CD, is, when a perpendicular AE is drawn from A the higheſt point of that line AB to the plane CD, and another line EB drawn from the point E, which the perpendicular AE makes in the Plane CD, to the end B of the ſaid line AB which is in the ſame Plane, whereby the angle is acute ABE which is contained under the inſiſting line AB, and the line drawn in the plane EB.



VI. The inclination of a Plane AB, to a Plane CD, is an acute angle FHG contained under the right lines FH, GH, which being drawn in either of the Planes AB, CD, to the ſame point H of the common ſection BE, make right angles FHB, GHB, with the common ſection BE.

VII. Planes are ſaid to be inclined to other Planes in the ſame manner, when the ſaid angles of inclination are equal one to another.

VIII. Parallel Planes are thoſe which being prolonged never meet.

IX. Like ſolid figures are ſuch as are contained under like Planes equal in number.

X. Equal and like ſolid figures are ſuch as are contained under like Planes equal both in multitude and magnitude.

XI. A ſolid angle is the inclination of more than two right lines which touch one another, and are not in the ſame ſuperficies.

Or thus;

A ſolid angle is that which is contained under more than two plane angles not being in the ſame ſuperficies, but conſiſting all at one point.

XII. A Pyramide is a ſolid figure comprehended under divers planes ſet upon one plane, (which is the baſe of the Pyramide) and gathered together to one point.

XIII. A Priſme is a ſolid figure contained under planes, whereof the two oppoſite are equal, like, and parallel; but the others are parallelograms.

XIV. A Sphere is a ſolid figure made when the diameter of a circle abiding unmoved, the ſemicircle is turned round about, till it return to the ſame place from whence it began to be moved.

Coroll.

Hence, all the rayes drawn from the center to the ſuperficies of a ſphere, are equal amongſt themſelves.

XV. The Axis of a ſphere, is that fixed right line, about which the ſemicircle is moved.

XVI. The Center of a ſphere, is the ſame point with that of the ſemicircle.

XVII. The Diameter of a ſphere, is a right line drawn thro' the center, and terminated on either ſide in the ſuperficies of the ſphere.

XVIII. A Cone is a figure made, when one ſide of a rectangled triangle (*viz.* one of thoſe that contain the right angle) remaining fixed, the triangle is turned round about till it return to the place from whence it firſt moved. And if the fixed right line be equal to the other which contains the right angle, then the Cone is a rectangled Cone: but if it be leſs, it is an obtuſe-angled Cone; if greater an acute-angled Cone.

XIX. The Axis of a Cone is that fix'd line about which the triangle is moved.

XX. The Baſe of a Cone is the circle, which is deſcribed by the right line moved about.

XXI. A Cylinder is a figure made by the moving round of a right-angled parallelogram, one of the ſides thereof, (namely which contain the right angle) abiding fix'd, till the parallelogram be turned about to the ſame place, where it began to move.

XXII. The Axis of a Cylinder is that quieſcent right line, about which the parallelogr. is turned.

XXIII. And the Baſes of a Cylinder are the circles which are deſcribed by the two oppoſite ſides in their motion.

XXIV. Like Cones and Cylinders, are they, both whoſe Axes and Diameters of their Baſes are proportional.

XXV. A Cube is a ſolid figure contained under ſix equal ſquares.

XXVI. A Tetraedron is a ſolid figure contained under four equal and equilateral triangles.

XXVII. An Octaedron is a ſolid figure contained under eight equal and equilateral triangles.

XXVIII. A Dodecaedron is a ſolid figure contained under twelve equal, equilateral and equiangular Pentagones.

XXIX. An Icoſaedron is a ſolid figure contained under twenty equal and equilateral triangles.

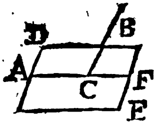
XXX. A Parallelepipedon is a ſolid figure contained under ſix quadrilateral figures, whereof thoſe which are oppoſite are parallel.

XXXI. A ſolid figure is ſaid to be inſcribed in a ſolid figure, when all the angles of the figure inſcribed are comprehended either within the angles, or in the ſides, or in the planes of the figure wherein it is inſcribed.

XXXII. Likewiſe a ſolid figure is then ſaid to be circumſcribed about a ſolid figure, when either the angles, or ſides, or planes of the circumſcribed figure touch all the angles of the figure which it contains.

PROP.

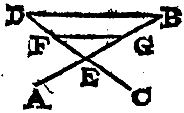
PROP. I.



One part AC of a right line cannot be in a plane ſuperficies, and another part CB elevated upward.

Produce AC in the plane directly to F. If you conceive CB to be drawn ſtrait from AC, then two right lines AB, AF, have one common ſegment AC. *a Which is impoſſible.* a 10.ax.1.

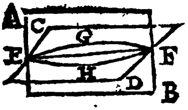
PROP. II.



If two right lines AB, CD, cut one another, they are in the ſame plane: And every triangle DEB is in one and the ſame plane.

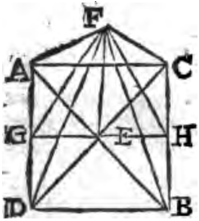
For imagine EFG, part of the triangle DEB, to be in one plane, and the part FDGB to be in another, then EF part of the right line ED is in a plane, and the other part elevated upwards; *a Which is abſurd.* Therefore the triangle EDB a 1. 11. is in one and the ſame plane; and ſo alſo are the right lines ED, EB; *a* wherefore the whole lines AB, DC, are in one plane. *Which was to be demonſtrated.*

PROP. III.



* *If two planes AB, CD, cut one the other, their common ſection EF is a right line.*

If EF the common ſection be not a right line, *a* then in the plane AB draw the right line EGF, *a* and a 1.*poſt.*1. in the plane CD draw the right line EHF. therefore two right lines EGF, EHF, include a ſuperficies. *b Which is abſurd.* b 14.*ax.*1.

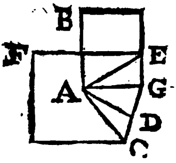


If a right line EF be at right angles erected upon two lines AB, CD, cutting one the other, at the common ſection E; it ſhall alſo be at right angles to the plane ACBD drawn by the ſaid lines.

Take EA, EC, EB, ED, equal one to the other, and
join the right lines AC, CB, BD, AD, draw
any right line GH thro' E, and join FA, FC,
a conſtr. FD, FB, FG, FH. Becauſe AE is *a* = EB, and
b 15. 1. DE *a* = EC, and the angle AED *b* = CEB. *c*
c 4. 1. therefore AD is = CB, *c* and likewiſe AC =
d ſch.34.1. DB. *d* therefore AD is parallel to CB, *d* and
e 29. 1. AC to DB. *e* wherefore the angle GAE=EBH,
f conſtr. and the angle AGE=EHB. But alſo AE *f* =EB.
g 26. 1. *g* therefore GE=EH, *g* and AG=BH. whence
h 4. 1. by reaſon of the right angles, by the hyp. and
ſo equal, at E, *h* the baſes FA, FC, FB, FD, are
equal. Therefore the triangles ADF, FBC, are
k 8. 1. equilateral one to another. *k* and thence the angle DAF=BCF. Therefore in the triangles AGF,
l 4. 1. FBH, the ſides FG, FH *l* are equal; and ſo by
conſequence the triangle FEG and FEH are
m 8. 1. mutually equilateral. *m* therefore the angles
n 10.def.1. FEG, FEH are equal, and *n* ſo right angles.
In like manner, FE makes right angles with all
o 3.def.11. the lines drawn thro' E in the plane ADBC, *o*
and is therefore perpendicular to the ſaid plane.

PROP.

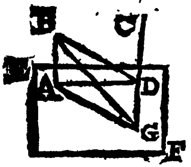
PROP. V.



If a right line AB be erected perpendicular to three right lines AC, AD, touching one the other at the common ſection, thoſe three lines are in the ſame plane.

For AC, AD, *a* are in one a 2. 11.
plane FC; *a* and AD, AE, are
in one plane BE. which if you conceive to be
ſeveral planes, then let their interſection *b* be b 3. 11.
the right line AG; therefore becauſe BA by
the Hypoth. is perpendicular to the right lines
AC, AD. *c* and ſo to the plane FC, *d* it is alſo c 4. 11.
perpendicular to the right line AG. therefore d 3. *def.* 11.
(ſince *a* that AB is in the ſame plane with AC,
AE) the angles BAG, BAE, are right angles,
and conſequently equal, the part and the whole.
Which is abſurd.

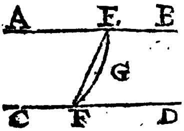
PROP. VI.



If two right lines AB, DC, be erected perpendicular to one and the ſame plane EF; thoſe right lines AB, DC are parallel one to the other.

Draw AD, whereunto let
DG=AB be perpendicular in
the plane EF, and join BD, BG, AG. Being in the
triangles BAD, ADG, the angles BAD, ADG *a* a *hyp.*
are right angles, and AB *b* = DG, and AD is b *conſtr.*
common, *c* therefore BD is = AG. whence in the c 4. 1.
triangles AGB, BGD, equilateral one to the o-
ther, the angle BAG is *d* = BDG; of which d 8. 1.
being BAG is a right angle, BDG ſhall be ſo
alſo. but the angle GDC is ſuppoſed right, there-
fore the right line GD is perpendicular to the
three lines DA, DB, CD. *e* which are therefore e 5. 11.
in the ſame plane *f* wherein AB is. Wherefore f 2. 11.
ſince AB and CD are in the ſame plane, and the
internal angles BAD, CDA, are right angles, *g* g 28. 1.
AB and CD ſhall be parallels. *Which was to be
dem.*

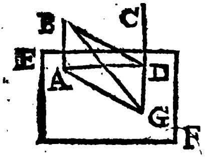
PROP. VII.



If there be two parallel right lines AB, CD, and any points E, F, be taken in both of them, the line EF which is joined at theſe points, is in the ſame plane with the parallels ABCD.

Let the plane in which AB, CD are, be cut
by another plane at the points E, F. then if EF
is not in the plane ABCD, it ſhall not be the
common ſection. Therefore let EGF be the
a 3. 11. common ſection; which *a* then is a right line.
b 14. *ax.* 1. therefore two right lines EF, EGF, include a
ſuperficies. *b Which is abſurd.*

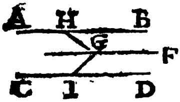
PROP. VIII.



If there be two parallel right lines AB, CD, whereof one AB is perpendicular to a plane EF. then the other CD ſhall be perpendicular to the ſame plane EF.

The preparation and demonſtration of the ſixth of this Book being
transfer'd hither; the angles GDA, and GDB
a 4. 11. are right angles: *a* therefore GD is perpendicu-
b 7. 11. lar to the plane, wherein are AD, DB (*b* in
c 3. *def.* 11. which alſo AB, CD are,) *c* therefore GD is
perpendicular to CD. but the angle CDA is alſo
d 29. 1. *d* a right angle. *e* therefore CD is perpendicular
e 4. 11. to the plane EF. *Which was to be dem.*

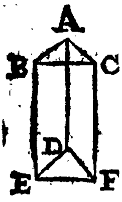
PROP. IX.



Right lines (AB, CD) which are parallel to the ſame right line EF, but not in the ſame plane with it, are alſo parallel one to the other.

In the plane of the parallels AB, EF, draw HG perpendicular to EF; alſo in the plane of the parallels EF, CD, draw IG perpendicular to
EF. *a* therefore EG is perpendicular to the plane a 4. 11.
wherein HG, GI are; and AH, CI are perpen- b 8. 11.
dicular to the ſame plane. *c* therefore EH and c 6. 11.
CI are parallels. *Which was to be dem.*

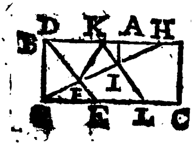
PROP. X.



If two right lines AB, AC, touching one another be parallel to two other right lines ED, DF, touching one another, and not being in the ſame plane, thoſe right lines contain equal angles, BAC, EDF.

Let AB, AC, DE, DF, be equal one to the other, and draw AB, BC, EF, BE, CF. Being AB, DE, *a* are
parallels, and equal, *b* alſo BE, AD, are paral- a *hyp. and*
lels and equal. In like manner CF, AD, are pa- *conſtr.*
rallels and equal; *c* therefore alſo BE, FC, are b 33. 1.
parallels and equal. *d* Therefore BC, EF are e- c 2. *ax.* 1.
qual. Wherefore ſince the triangles BAC, EDF, *and* 30. 1.
are of equal ſides one to the other, the angles d 33. 1.
BAC, EDF *e* ſhall be equal. *Which was to be dem.* e 8. 1.

PROP. XI.



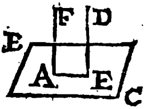
From a point given on high A, to draw a right line AI perpendicular to a plane below BC.

In the plane BC draw any

a 12. 1. line DE; to which from the point A *a* draw the
b 11. 1. perpendicular AF, and *b* likewiſe FH in the
plane BC cutting the ſaid line DE at F; *a* then
let fall AI perpendicular to FH. Which AI
ſhall be perpendicular to the plane BC.

c 31. 1. For thro' I *c* let KIL be drawn parallel to DE.
d *conſtr.* Becauſe DE *d* is perpendicular to AF, and FH, *e*
e 4. 11. therefore DE ſhall be perpendicular to the plane
f 8. 21. IFA. and ſo alſo KL *f* is perpendicular to the
g 3.*def.*11. ſame plane. *g* therefore the angle KIA is a right
h *conſtr.* angle. but the angle AIE is alſo *h* a right angle.
l 4. 11. *l* therefore AI is perpendicular to the plane BC.
Which was to be done.

PROP. XII.

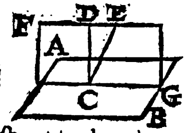


In a plane given BC, at a point given therein A, to erect a perpendicular line AF.

From ſome point without the
a 11. 11. plane D, *a* draw DE perpendi-
cular to the ſaid plane BC, and joining the
b 31. 1. points A, E, by a line AE, *b* draw AF parallel
c 8. 11. to DE. *c* it is apparent that AF is perpendicular to the plane BC. *Which was to be done.*

This and the preceding Problem are practically performed by applying two ſquares to the point given; as appears by 4. 11.

PROP. XIII.

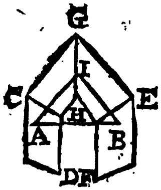


As a point given C in a plane given AB, two right lines CD, CE, cannot be erected perpendicular on the ſame ſide.

a 6. 11. For both CD, and CE, *a*
ſhould then be perpendicular to the plane AB,
and conſequently parallels; which is repugnant
to the definition of parallel lines.

PROP.

PROP. XIV.

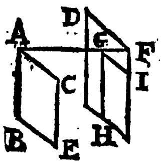


Planes CD, FE, to which the ſame right line AB is perpendicular, are parallel.

If you deny this; then let the planes CD, FE, meet, ſo that their common ſection be the right line GH, in which take any point I, draw to it the right lines IA, IB, in the ſaid planes, whereby in the triangle IAB, two angles IAB, IBA *a* are right angles. *b Which is abſurd.*

a *hyp.* and 3. *def.* 11.
b 17. 1.

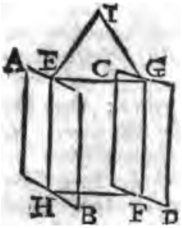
PROP. XV.



If two right lines AB, AC, touching one the other, be parallel to two other right lines DE, DF, touching one the other, and not being in the ſame plane with them, the planes BAC, EDF, drawn by thoſe right lines are parallel one to the other.

From A *a* draw AG perpendicular to the a 11. 11.
plane EF. *b* and let GH, GI be parallel to DE, b 31. 1.
DF. *c* theſe alſo ſhall be parallel to AB, AC. c 30. 1.
Therefore ſince the angles IGA, HGA, *d* are d 3.*def.*11.
right angles, alſo CAG, BAG, *e* ſhall be right e 29. 1.
angles. *f* therefore GA is perpendicular to the f 4. 11.
plane BC; but the ſame is perpendicular to the g *conſtr.*
plane EF. *h* therefore the planes BC, EF, are h 14. 11.
parallel. *Which was to be dem.*

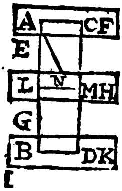
PROP. XVI.



If two parallel planes AB, CD, be cut by ſome other plane HEIGF, their common ſections EH, GF are parallel one to the other.

For if they be conceived to be otherwiſe; being in the ſame plane that cuts them, they will meet ſomewhere, if produced; ſuppoſe in I; wherefore
a 1. 11. ſince the whole lines HEI, FGI *a* are in the planes AD, CD, being produced, the planes alſo ſhall meet. *contrary to the Hyp.*

PROP. XVII.

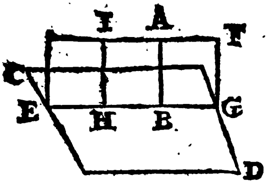


If two right lines ALB, CMD, be cut by parallel planes EF, GH, IK; they ſhall be cut proportionally, (AL. LB :: CM. MD.

Let the right lines AC, BD, be drawn in the planes EF, IK; as alſo AD paſſing thro' the plane GH in the point N. and join NL, LM the planes of the triangles ADC, ADB,
a 16. 11. make the ſections DB, LN, and AC, NM *a*
b 2. 6. parallels. Therefore AL. LB :: AN. ND *b* :: CM. MD. *Which was to be dem.*

PROP.

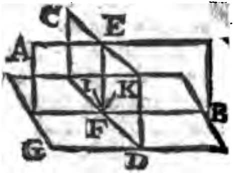
PROP. XVIII.



If a right line AB be perpendicular to ſome plane CD, all the planes extended by that right line AB (EF,&c.) ſhall be perpendicular to the ſame plane CD.

Let there be ſome
plane EF drawn by AB, making the ſection EG a 31. 1.
with the plane CD; from ſome point whereof b 8. 11.
H, *a* draw HI parallel to AB in the plane EF; c 4. *def.* 11.
b then ſhall HI be perpendicular to the plane CD, and ſo likewiſe any other lines, that are perpendicular to EG. *b* therefore the plane EF is perpendicular to the plane CD; and by the ſame reaſon any other planes drawn by AB ſhall be perpendicular to EF. *Which was to be dem.*

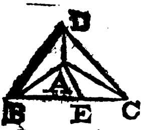
PROP. XIX.



If two planes AB, CD, cutting one the other, be perpendicular to ſome plane GH. their line of common ſection EF ſhall be perpendicular to the ſame plane (GH.)

Becauſe the planes AB, CD, are taken perpendicular to the plane GH, it appears by 4. def. 11. a 13. 11.
that out of the point F there may be drawn in both planes AB, CD, a perpendicular to the plane GH. which ſhall be *a* but one; and therefore the common ſection of the ſaid planes. *Which was to be dem.*

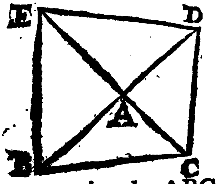
PROP. XX.



If a ſolid angle ABCD be contained under three plane angles, BAD, DAC, BAC, any two of them howſoever taken are greater than the third.

If the three angles are equal, the aſſertion is
evident; if unequal, then let the greateſt be BAC;
from whence *a* take away BAE = BAD, and
a 23. 1. make AD = AE; and alſo draw BEC, BD, DC.
Becauſe the ſide BA is common, and AD *b* =
b *conſtr.* AE; and the angle BAE *b* = BAD. *c* thence is
c 4. 1. BE = BD. but BD + DC is *d* $\sqsubset$ BC. *e* therefore
d 20. 1. DC $\sqsubset$ EC. Wherefore ſince AD *b* = AE, and the
e 5. *ax.* 1. ſide AC is common, and DC $\sqsubset$ EC. *f* the angle
f 25. 1. CAD ſhall be $\sqsubset$ EAC. *g* therefore the angle
g 4. *ax.* 1. BAD + CAD $\sqsubset$ BAC. *Which was to be dem.*

PROP. XXI.

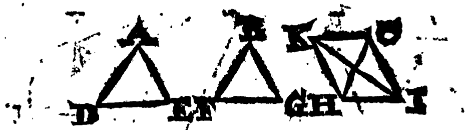


Every ſolid angle A is contained under leſs angles than four plane right angles.

For let a plane any-wiſe cutting the ſides of the ſolid angle A make a many-ſided figure BCDE, and as many triangles ABC, ACD, ADE, AEB. I denote all the angles of the polygone by X; and
I term the ſum of the angle at the baſes of the
a 32. 1. & triangles Y. wherefore X + 4 right angles *a* =
ſch. 31. 1. Y + A. but being that, (of the angles at B) *b*
b 20. 11. the angle ABE + ABC is $\sqsubset$ CBE, and the ſame
c 5. *ax.* 1. is true alſo of the angles at C, at D, and at E,
c it is manifeſt that Y is $\sqsubset$ X. and conſequently A ſhall be $\sqsupset$ 4 right angles. *Which was to be dem.*

PROP.

PROP. XXII.



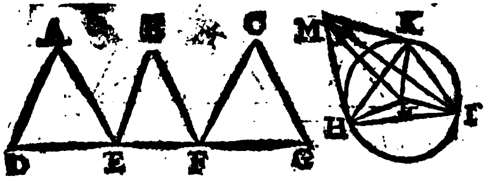
If there be three plane angles A, B, HCI, whereof two howſoever taken are greater than the third, and the right lines which contain them be equal AD, AE, FB, &c. then of the right lines DE, FG, HI, coupling thoſe equal right lines together, it is poſſible to make a triangle.

A triangle may be *a* made of them, if any two be greater than the third: but they are ſo. For *b* make the angle HCK = B, and CK = CH, and draw HK, IK *c* thence HK=FG. and becauſe the angle KCI *d* ⊏ A. *e* therefore KI ⊏ DE. but KI *f* ⊐ HI + KH (FG.) therefore DE ⊐ HI + FG. By the like argument any two may be proved greater than the third; and conſequently *a* it is poſſible to make a triangle of them. *Which was to be dem.*

a 22. 1.
b 23. 1.
c 4. 1.
d *hyp.*
e 24. 1.
f 20. 1.

PROP.

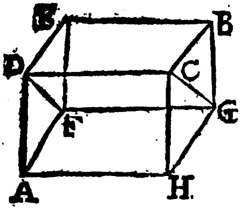
PROP. XXIII.



To make a ſolid angle MHIK of three plane angles A, B, C, whereof two howſoever taken are greater
* 21. 11. *than the third.* * *But it is neceſſary that thoſe three angles be leſs than four right angles.*

Make AD, AE, BE, BF, CF, CG, equal one
to the other; and of the ſubtended lines DE, EF,
a 22. 11. FG (that is, of the equal lines HI, IK, KH) *a*
and 22. 1. make the triangle HKI; about which *b* deſcribe
b 5. 4. the circle LHKI. * But becauſe AD is $\sqsubset$ HL,
* *See Cla- c* let ADq be $=$ HLq $+$ LMq. *d* and let LM
vius. be perpendicular to the plane of the circle HKI,
c *ſch.*47.1. and draw HM, KM, IM. wherefore ſince the
d 12. 11. angle HLM *e* is a right angle, *f* thence is MHq
e 3.*def.*11. $=$ HLq $+$ LMq *g* $=$ ADq. therefore MH $=$
f 47. 1. AD. By the ſame reaſon MK, MI, AD (that is,
g *conſtr.* AE, EB, *&c.*) are equal; therefore ſince HM $=$
h *conſtr.* AD, and MI$=$AE, and DE $=$ HI. *k* the angle
k 8. 1. A ſhall be $=$ HMI, *k* as likewiſe the angle IMK
$=$ B, *k* and the angle HMK $=$ C. wherefore a
ſolid angle is made at M of the three given
plane angles. *Which was to be done.* AD is aſſumed to be $\sqsubset$ HL. But this is manifeſt. For if
l *conſtr.&* AD be $=$ or $\sqsupset$ HL, then is the angle A *l* $=$
8. 1. *m* or $\sqsubset$ HLI. In like manner ſhall B be $=$ or $\sqsubset$
m 22. 1. HLK, and C $=$ or $\sqsubset$ KLI. wherefore A$+$B$+$C
* 4.*cor.*13. * ſhall either equal or exceed four right angles,
1. *contrary to the Hypoth.* therefore rather let AD
be $\sqsubset$ HL. *Which was to be dem.*

PROP. XXIV.

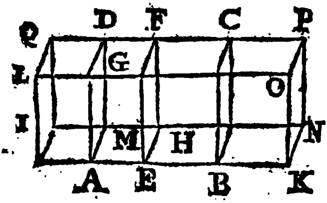


If a ſolid AB be contained under parallel planes, the oppoſite planes thereof (AG, DB, &c.) are like and equal parallelograms.

The plane AC cutting the parallel planes AG, DB, *a* makes the ſections AH, DC, parallels. and by the ſame reaſon AD, HC are parallels. Therefore ADCH is a parallelogram. By the like argument the other planes of the parallelepipedon are *b* parallelograms, wherefore being AF is parallel to HG, and AD to HC, *c* the angle FAD ſhall be = CGH. therefore becauſe AF *d* = HG, and AD *d* = HC, and ſo AF. AD :: HG. HC, the triangles FAD, GHC, *g* are like and *h* equal; and conſequently the parallelograms AE, HB are like and *k* equal. and the ſame may be ſhewn of the other oppoſite planes. therefore, *&c.*

a 16. 11.
b 35. *def.* 1.
c 10. 11.
d 34. 2.
e 7. 5.
g 6. 6.
h 4. 1.
k 6. *ax.* 1.

PROP. XXV.



If a ſolid Parallelepipedon ABCD be cut by a plane EF, parallel to the oppoſite planes AD, BC; then as the baſe AH is to the baſe BH, ſo ſhall ſolid AHD be to ſolid BHC.

Conceive the Parallelepipedon to be extended on either ſide, and take AI = AE, and BK = EB, and put the plane IQ, KP, parallel to the planes AD, BC; then the pgrs. IM, AH, and *a* DL, DG, *b* and IQ, AD, EF, *&c.* are *a* like and equal. *c* wherefore the Parallelepipedon AQ is = AF; and by

a 36. 1. & 1. *def.* 6.
b 24. 11.
c 10. *def.* 11.

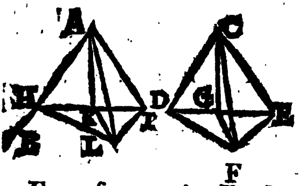
by the ſame reaſon the Parallelepipedon BP = BF. therefore the ſolids IF, EP are as multiple of the ſolids AF, EC, as the baſes IH, KH, are of the baſes AH, BH. And if the baſis IH be
d 24. 11. ⊏, =, ⊐ KH, *d* likewiſe ſhall the ſolid IE be
and 9. *def.* ⊏, =, ⊐ EP. *e* conſequently AH. BH :: AF.
11. EC. *Which was to be dem.*

e 6. *def.* 5. *The ſame may be accommodated to all ſorts of Priſms, whence*

Coroll.

If any Priſm whatſoever be cut by a plane parallel to the oppoſite planes, the ſection ſhall be a figure equal and like to the oppoſite planes.

PROP. XXVI.

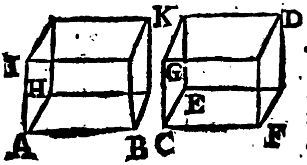


Upon a right line given AB, and at a point given in it A, to make a ſolid angle AHIL equal to a ſolid angle given CDEF.

a 11. 11. From ſome point F *a* draw FG perpendicular to the plane DCE, and draw the right lines DF, FE, EG, GD, CG. Make AH = CD, and the angles HAI = DCE, and AI = CE; and in the plane HAI make the angle HAK = DCG, and AK = CG. then erect KL perpendicular to the plane HAI, and let KL be = GF. and draw AL: then AHIL ſhall be a ſolid angle equal to that given CDEF. For the conſtruction of this does wholly reſemble the framing of that, as may eaſily appear to any that examine it.

PROP.

PROP. XXVII.

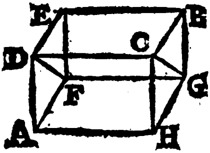


Upon a right line given AB to deſcribe a parallelepipedon AK, like, and in like manner ſituate, with a ſolid parallelepipedon given CD.

Of the plane angles BAH, HAI, BAI, which
are equal to FCE, ECG, FCG, *a* make the ſolid a 26. 11.
angle A equal to the ſolid angle C: alſo *b* make b 12. 6.
FC. CE :: BA. AH. *b* and CE. CG :: AH. AI (*c* c 22. 5.
whence of equality FC.CG :: BA. AI) and finiſh
the parallelepipedon AK, which ſhall be like to
that which is given.

For by the conſtruction, the Parallelogram *d* d 1. *def.* 6.
BH is like FE, and *d* HI to EG, and *d* BI to FG,
and *e* ſo the oppoſites of theſe to the oppoſites e 14. 11.
of them: therefore the ſix planes of the ſolid
AK are like to the ſix planes of the ſolid CD,
f and conſequently AK, CD, are like ſolids. f 9. *def.* 11.
Which was to be dem.

PROP. XXVIII.

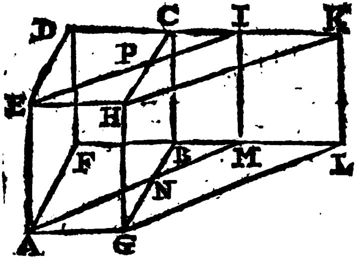


If a ſolid parallelepipedon AB be cut by a plane FGCD drawn by the diagonal lines DF, CG, of the oppoſite planes AE, HB, that ſolid AB ſhall be equally biſected by the plane FGCD.

For becauſe DC, FG, are *a* equal and parallels, a 24. 11.
b the plane FGCD is a Pgr. and being *a* the Pgrs. b 34. 1.
AE, HB, are equal and like, *b* alſo the triangles
AFD, HGC, CGB, DFE are equal and like. But
the Pgrs. AC, AG, are equal and like to FB and
FD. therefore all the planes of the priſme FGC-
DAH are equal and like to all the planes of the
priſme FGCDEB, and *c* conſequently this priſme c 9. *def.* 11.
is equal to that. *Which was to be dem.*

PROP.

PROP. XXIX.



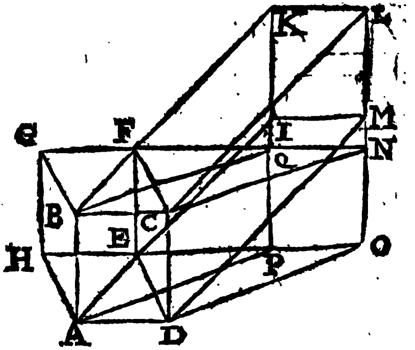
*Solid Parallelepipedons AGHEFBCD, AGHEMLKI, being conſtituted upon the ſame baſe AGHE, and * in the ſame height, whoſe inſiſting lines AF, AM, are placed in the ſame right lines AG, FL, are equal one to the other.*

For a if from the equal priſmes AFMEDI, GBLHCK, the common priſme NBMPCI be taken away, and the ſolid AGNEHR be added, the Parallelepipedon AGHEFBCD ſhall be = AGHEMLKI. *Which was to be dem.*

** i. e. between the parallel planes AGHE, FLKD, and ſo underſtand it in the following.*

a 10. *def.* 11 *and* 35. 1.
b 3. *and* 1. *ax.* 1.

PROP. XXX.

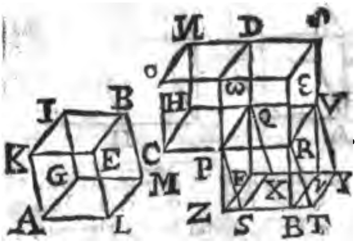


Solid parallelepipedons ADBCHEFG, ADCBIMLK being conſtituted upon the ſame baſe ADBC, and in *in the ſame height, whoſe inſiſting lines AH, AI are not placed in the ſame right lines, are equal one to the other.*

For produce the right lines HEO, GFN, and LMO, KIP; and draw AP, DO, BQ, CN. *a* then ſhall DC, AB, HG, EF, PQ, ON be as well equal and parallel one to the other as AD, HE, GF, BC, KL, IM, QN, PO. *b* wherefore the parallelepipedon ADCBPONQ ſhall be equal to either parallelepipedon ADCBHEFG, ADCBIMLK; and *c* conſequently theſe two are equal one to the other. *Which was to be dem.*

a 34. 1.
b 29. 11.
c 1. *ax.* 1.

PROP. XXXI.



Solid parallelepipedons, ALEKGMBI, CPωOHQDN, being conſtituted upon equal baſes ALEK, CPωO, and in the ſame height are equal, one to the other.*

* *by height underſtand the perpendicular drawn from the plane of the baſe to the oppoſite plane.*

Firſt, let the parallelepipedons AB, CD, have the ſides perpendicular to the baſes, and at the ſide CP being produced, *a* make the parallelogram PRTS equal and like to the parallelogram KELA. *b* and ſo the parallelepipedon PRTSQVYX equal and like to the parallelepipedon AB. Produce OωE, NDδ, ωPZDQF, ERB, δVγ, TSZ, YXF, and draw Eδ, Bγ, ZF.

a 18. 6.
b 27. 11. *and* 10. *def.* 11.

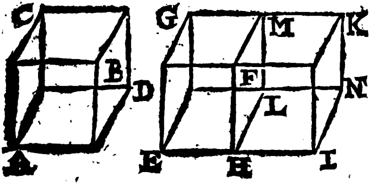
The planes OωδN, CRVHZTYF *c* are parallels one to the other; *d* and the pgrs. ALEK, CDωO, PRTS, PRBZ are equal. Therefore ſince the parallelepipedon CD.PVδω :: pgr. Cω (PRBZ) Ps *e* :: parallelepipedon PRBZQVγF.BVδω; the paral-

c 30. *def.* 11.
d *hyp. and* 35. 1.
e 25. 11.

f 9. 5. parallelepipedon CD *ſ*ſhall be = PRBZVQγF *g* =
g 29. 11. PRVQSTYX *h* = AB. *Which was to be dem.*

h *conſtr.* But if the parallelepipedons AB, CD, have
ſides oblique to the baſes, then on the ſame baſes and in the ſame heighth place parallelepipe-
k 29. 11. dons whoſe ſides are perpendicular to the baſe.
k They ſhall be equal to one another, and thoſe
m 1. *ax.* 1. that are oblique; *m* whence alſo the oblique
parallelepipedons AB, CD are equal. *Which was to be dem.*

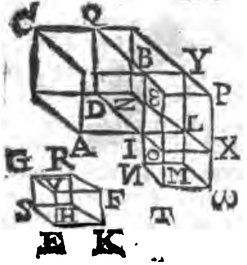
PROP. XXXII.



Solid parallelepipedons ABCD, EFGL, of the ſame heighth, are one to the other, as their baſes AB, EF.

a 45. 1. Produce EHI, *a* and make the pgr. FI = AB,
b 31. 1. and *b* compleat the parallelepipedon FINM. It
c 31. 11. is clear that the parallelepipedon FINM. (c
d 25. 11. ABCD.) EFGL *d* :: FI (AB.) EF. *Which was to be dem.*

PROP. XXXIII.



Like ſolid parallelepipedons, ABCD, EFGH, are in tripled proportion one to the other of that in which their homologous ſides or of like proportion AI, EK, are.

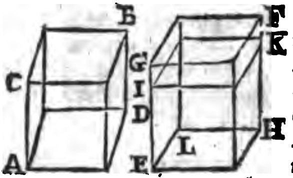
Produce the right lines
a 31. 1. AIL, DIO, BIN, and *a*
make IL, IO, IN, equal
to

to EK, KH, KF, *b* and ſo the parallelepipedon b 27. 11.
IXMT equal and like to the parallelepipedon c 31. 1.
EFGH. *c* Let the paralleps. IXPB, DLYQ be fi- d *hyp.*
niſhed. *d* Then ſhall be AL. IL (EK) :: DI. IO e 1. 6.
(HK) :: BI. IN. KF. *e* that is, the Pgr. AD. DL f 32. 11.
:: DL. IX :: BO. IT. *f i. e.* the parallepp. ABCD. g *conſtr.*
DLQY :: DLQY. IXBP :: IXBP. IXMT. (*g* EFGH.) h 10. *def.* 5.
h therefore the proportion of ABCD to EFGH k 1. 6.
is triple of the proportion of ABCD to DLQY,
k or of AI to EK. *Which was to be dem.*

Coroll.

Hence it appears, that if four right lines be continually proportional, as the firſt is to the fourth, ſo is a parallelepipedon deſcribed on the firſt to a parallelepipedon deſcribed on the ſecond, being like and in like manner deſcribed.

PROP. XXXIV.



In equal ſolid parallelepipedons ADCB, EHGF, the baſes and altitudes are reciprocal (AD. EH :: EG. AC.) And ſolid parallelepipedons, ADCB, EHGF, whoſe baſes and altitudes are reciprocal, are equal.

Firſt, let the ſides CB, GE be perpendicular to a 3. 1.
the baſes; then if the altitudes of the ſolids are b 31. 1.
equal, the baſes alſo ſhall be equal, and the thing c 32. 11.
is clear. But if the altitudes are unequal, from d 17. 5.
the greater EG *a* take EI = AC, and at I *b* draw e 1. 6.
the plane IK parallel to the baſe EH. then f *conſtr.*
1. *Hyp.* AD. EH *c* :: parallepp. ADCB. EHIK g 11. 5.
d :: parallepp. EHGF. EHIK *c* :: GL. IL *e* :: GE. h 32. 11.
IE. (*f* AC.) *g* it is plain therefore that AD. EH k *hyp.*
:: GE. AC. *Which was to be dem.* l 1. 6.
2. *Hyp.* ADCB. EHIK *b* :: AD. EH *k* :: EG. EI *l* m 32. 11.
:: GL. IL *m* :: parallepp. EHGF. EHIK. *n* where- n 9. 5.

T fore

ſore the parallelepipedon ADCB = EHGF. *Which was to be dem.*

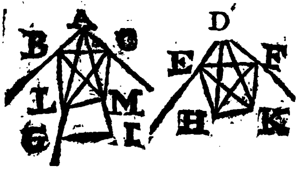
Moreover, let the ſides be oblique to the baſes, and erect right parallelepipedons upon the ſame baſes in the ſame altitude; the oblique parallelepipedons ſhall be equal to them. Wherefore ſince by the firſt part, the baſes and altitudes of thoſe be reciprocal, the baſes and altitudes of theſe alſo ſhall be reciprocal. *Which was to be dem.*

Coroll.

All that hath been demonſtrated of parallelepipedons in the 29,30,31,32,33,34 Prop. does alſo agree in triangular priſmes, which are half parallelepipedons, as appears by Prop. 28. Therefore,

1. Triangular priſmes are of equal heighth with their baſes.
2. If they have the ſame or equal baſes and the ſame altitude, they are equal.
3. If they be like, their proportion is triple to that of their ſides of like proportion.
4. If they be equal, their baſes and altitudes are reciprocal; and if their baſes and altitudes be reciprocal, they are alſo equal.

PROP. XXXV.



If there be two plane angles BAC, EDF, equal, and from the points of thoſe angles two right lines AG, DH be elevated on high, containing equal angles with the lines firſt given, each to his correſpondent angle (the angle GAB = HDE, and GAC = HDF.) and if in thoſe elevated lines AG, DH, ſome points be taken G, H; and from theſe points perpendicular lines GI, HK, drawn to the planes BAC, EDF, in which the angles firſt given are, and right lines

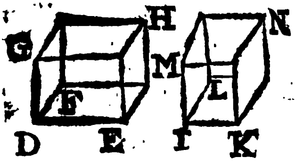
lines AI, DK, be drawn to the angles firſt given from the points I K, which are made by the perpendiculars in the planes; thoſe right lines with the elevated lines AG, DH ſhall contain equal angles GAM, HDK.

Make DH, AL, equal; and GI, LM parallels, and MC to AC, MB to AB, KF to DF, KE to DE perpendicular; and draw the right lines BC, LB, LC, and EF, HF, HE; *a* and LM is per- a 8. 11.
pendicular to the plane BAC; *b* wherefore the b 3. *def.* 11.
angles LMC, LMA, LMB; and by the ſame reaſon the angles HKF, HKD, HKE are right angles. Therefore ALq *c* = LMq + AMq *c* = c 47. 1.
LMq + CMq + ACq *c* = LCq + ACq. *d* there- d 46. 1.
fore the angle ACL is a right angle. Again e 47. 1.
ALq *e* = LMq + MAq *e* = LMq + BMq + BAq *e* = BLq + BAq. *d* therefore the angle ABL is alſo a right angle. By the like inference the angles DFH. DEH are right angles; *f* therefore AB = DE, *f* and BL = EH, *f* and AC f 26. 1.
= DF, and CL = FH. *g* wherefore alſo BC = g 4. 1.
EF; *g* and the angle ABC = DEF, *g* and the angle ACB = DFE. whence the other right h 1. *ax.* 1.
angles CBM, BCM, are equal to the other FEK, k 26. 1.
EFK. *k* therefore CM = FK, *l* and ſo alſo AM l 47. 1.
= DK. therefore if from LAq *m* = HDq be m *conſtr.*
taken away AMq = DKq, *n* there remains n 47. 1. &
LMq = HKq. wherefore the triangles LAM, 3. *ax.*
HDK are equilateral one to the other; *o* o 8. 1.
therefore the angle LAM = HDK. *Which was to be dem.*

Coroll.

Therefore, if there be two plane angles equal, from whoſe points equal right lines be elevated on high, containing equal angles with the lines firſt given, each to each; perpendiculars drawn from the extreme points of thoſe elevated lines to the planes of the angles firſt given, are equal one to the other; *viz.* LM = HK.

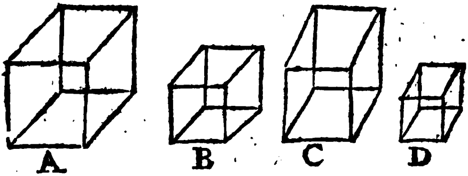
PROP. XXXVI.



If there be three right lines DE, DG, DF proportional, the ſolid parallelepipedon DH made of them, is equal to the ſolid parallelepipedon IN made of the middle DG (IL) which is alſo equilateral, and equiangular to the ſaid parallelepipedon DH.

a *hyp.* Becauſe DE. IK *a* :: IL. DF. *b* the parallelo-
b 14.6. gram LK ſhall be = FE. and by reaſon of the
equality of the plane angles at E and I, and of
the lines GD, IM, alſo the altitudes of the pa-
c 31.11. rallelepipedons are equal by the preceding Co-
rollary, therefore the parallelepipedons are equal
one to the other. *Which was to be dem.*

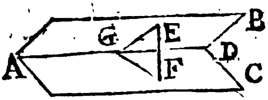
PROP. XXXVII.



If there be four right lines A, B, C, D, proportional, the ſolid parallelepipedons A, B, C, D being like, and in like ſort deſcribed from them ſhall be proportional. And if the ſolid parallelepipedons, being like and in like ſort deſcribed, be proportional (A. B :: C. D) then thoſe right lines A, B, C, D, ſhall be proportional.

For the proportions of the parallelepipedons
a 33.11. *a* are triple of thoſe of the lines; therefore if A.
b *ſch.*23.5. B :: C. D. *b* then ſhall the parallelepipedon A.
parallelepipedon B :: parallelepipedon C. parallelepipedon D. and ſo alſo contrarily.

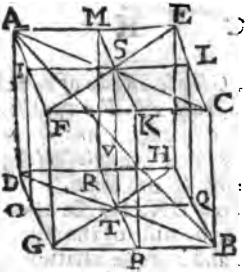
PROP. XXXVIII.



If a plane AB be perpendicular to a plane AC, and a perpendicular line EF be drawn from a point E in one of the planes (AB) to the other plane AC, that perpendicular EF ſhall fall upon the common ſection of the planes AD.

If it be poſſible, let F fall without the inter-
ſection AD, and in the plane AC *a* draw FG a 12. 1.
perpendicular to AD, and join EG. The angle
FGE *b* is a right angle, and EFG is ſuppoſed to b 4. *and* 3.
be ſuch alſo; therefore two right angles are in *def.* 11.
the triangle EFG. *c Which is abſurd.* c 17. 1.

PROP. XXXIX.



If the ſides (AE, FC, AF, EC, and DH, GB, DG, HB) of the oppoſite planes AC, DB, of a ſolid parallelepipedon AB, be divided into two equal parts and planes ILQO, PKMR, be drawn thro' their ſections, the common ſection of the planes ST, and the diameter of the ſolid parallelepipedon AB ſhall divide one the other into two equni parts. a 34. 1.

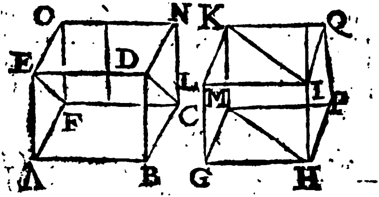
Draw the right lines SA, SC, TD, TB. Becauſe b 29. 1.
a the ſides DO, OT are equal to the ſides BQ, QT. c 4. 1.
b and the alternate angles TOD, TQB equal al- d *ſch.* 15. 1.
ſo, *c* the baſe *e* DT, TB, and the angles DTO, e 34. 1.
BTQ are equal. *d* therefore DTB is a right line, f 9. 11. &
and ſo in like manner is ASC. Moreover *e* as 1. *ax.*
well AD is parallel and equal to FG *e* as FG to g 33. 1.
CB, and *f* thence AD is parallel and equal to C, h 7. 11.

g 33. 1. *g* and conſequently AC to DB. *h* wherefore AB
h 7. 11. and ST are in the ſame plane ABCD. Therefore
ſince the vertical angles AVS, BVT, and the al-
k 7. *ax.* 1. ternate angles ASV, BTV are equal; *k* and AS
l 26. 1. = BT; therefore ſhall AV be = BV. *l* and SV
= VT. *Which was to be dem.*

Coroll.

Hence in every parallelepipedon all the diameters biſect one another in one point, V.

PROP. XL.



If two Priſms ABCFED, GHMLIK, be of equal altitude, whereof one hath its baſe ABCF a parallelogram, and the other GHM, a triangle; and if the parallelogram ABCF be double to the triangle GHM; theſe Priſms ABCFED, GHMLIK are equal.

For if the parallelepipedons AN, GQ, be com-
a 31. 11. pleted, *a* they ſhall be equal, becauſe of the equa-
b 34. 1. & lity *b* of the baſes AC, GP, and *c* of the altitudes.
7. *ax.* *d* therefore alſo the priſms, *e* the halfs thereof
c *hyp.* ſhall be equal. *Which was to be dem.*
d 28. 11.

Coroll.

e 7. *ax.* 1. *From the preceding demonſtrations, the dimenſion*
Andr. *of triangular Priſms, and quadrangular, or paralle-*
Tacq. *lepipedons, is learnt; viz. by multiplying the altitude into the baſe.*

As if the altitude be 10 feet, and the baſe 100 ſquare feet (the baſe may be meaſured by *ſch.* 35. 1. or by 41. 1.) then multiply 100 by 10, ... 1000

1000 cubic feet ſhall be produced, for the ſolidity of the priſm given.

For as a rectangle, ſo alſo is a right parallelepipedon produced of the altitude multiplied into the baſe. Therefore every parallelepipedon is produced of the altitude multiplied into the baſe, as appears by 31. of this Book.

Moreover, ſince the whole parallelepipedon is produced of the altitude drawn into the baſe, the half thereof (that is, a triangular Priſm) ſhall be produced of the altitude drawn into half the baſe, namely the triangle.

An Advertiſement.

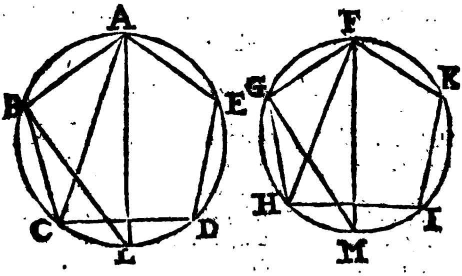
Obſerve, That of thoſe letters which denote a ſolid angle, the firſt is always at the point in which the angle is; but of thoſe letters which denote a Pyramide, the laſt is at the ſupreme point thereof.

Ex. gr. The ſolid angle ABCD is at the point A; and the ſupreme point of the Pyramide BCDA is at the point A. and the baſe is the triangle BCD.

The End of the eleventh Book.

THE TWELFTH BOOK OF EUCLIDE's ELEMENTS.

PROP. I.



Like the polygonous figures ABCDE, FGHIK, deſcribed in circles ABD, FGI, are one to another, as the ſquares deſcribed of the diameters of the circles AL, FM.

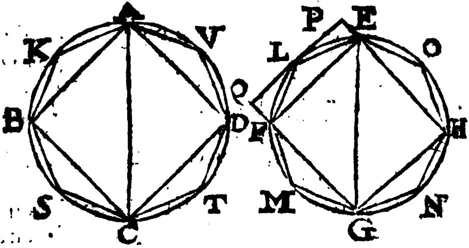
Draw AC, BL, FH, GM. Becauſe
a 1. *def.* 6. *a* the angle ABC = FGH, *a* and AB. BC :: FG.
b 6. 6. GH. *b* therefore ſhall the angle ACB (*c* ALB)
c 21. 3. be = FHG (*c* FMG.) but the angles ABL,
d 31. 3. FGM *d* are right and ſo equal; *e* therefore the
e 32. 3. triangles ABL, FGM are equiangular. *f* where-
f *cor.* 4. 6. fore AB. FG :: AL. FM. *g* therefore ABCDE,
g 22. 6. FGHIK :: ALq. FMq.

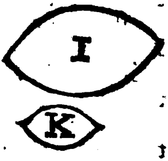
Coroll.

Hence (becauſe AB. FG :: AL. FM :: BC. GH,
&c.) the contents of like polygonous figures de-
h 1. 12. & ſcribed in a circle are in *h* proportion as the
12. 5. diameters.

PROP.

PROP. II.





Circles ABT, EFN, are in proportion one to another, as the ſquares of their diameters AC, EG are.

Suppoſe ACq. EGq :: the circle ABT. I, I ſay then I is equal to the circle EFN.

For firſt, if it be poſſible, let
I be leſs than the circle EFN, and let K be the
exceſs or difference. Inſcribe the ſquare EFGH
in the circle EFN, *a* it being the half of a cir- a *ſch.*7. 4.
cumſcribed ſquare, and ſo greater than the ſemi-
circle. *b* Divide equally in two the arches EF, b 30. 3.
FG, GH, HE, and at the points of the diviſions
join the right lines EL, LF, &c. at L draw the
tangent PQ (*c* which is parallel to EF) and pro- c *ſch.*27. 3.
duce HEP, GFQ. then is the triangle ELF *d* d 41. 1.
the half of the parallelogram EPQF, and ſo greater
than the half of the ſegment ELF; and in like
ſort the reſt of thoſe triangles exceed the halfs
of the reſt of the ſegments. And if the arches
EL, LF, FM, &c. be again biſected, and the right
lines joined, the triangles will likewiſe exceed
the half of the ſegments. Wherefore if the ſquare
EFGH be taken from the circle EFN, and the
triangles from the other ſegments, and this be
done continually, at length *e* there will remain e 1. 10.
ſome magnitude leſs than K. Let us have gone ſo
far, namely to the ſegments EL, LF, FM, &c.

taken

f *hyp.* *and* taken together leſs than K. Therefore I (*f* the
3. *ax.* circle FENK) ⊐ the polyg. ELFMNHO (the
g 30.3. & circle FEN — the ſegm. EL + LF, &c.) In the
1. *poſt.* 1. circle ABT *g* conceive a like polygonon AKBS-
h 1. 12. CTDV inſcribed. therefore ſince AKBSCTDV.
k *hyp.* ELFMGNHO *h* :: ACq, EGq *k* :: the circle
l 9. *ax.* 1. ABT. I. and the polyg. AKBSCTDV *l* ⊐ the
m 14. 5. circle ABT. the polyg. ELFMGNHO *m* ſhall
be ⊐ I. but before, I was ⊐ ELFMGNHO.
which is repugnant.

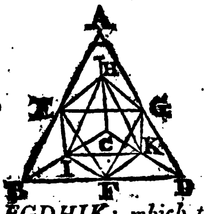
Again, if it be poſſible, let I be ⊏ the circle
n *hyp.* EFN. Therefore becauſe ACq.EGq *n* :: the circle
ABT. I; and inverſely I. the circle ABT. EGq.
ACq. ſuppoſe I. the circle ABT :: the circle
o 14. 5. EFN. K. *o* therefore the circle ABT ⊏ K. *p* and
p 11. 5. EGq. ACq :: the circle EFN. K. which is ſhewn
to be repugnant.

Therefore it muſt be concluded, that I is =
to the circle EFN. *Which was to be dem.*

Coroll.

Hence it follows, that as a circle is to a circle, ſo is a polygonon deſcribed in one to a like polygonon deſcribed in the other.

PROP. III.

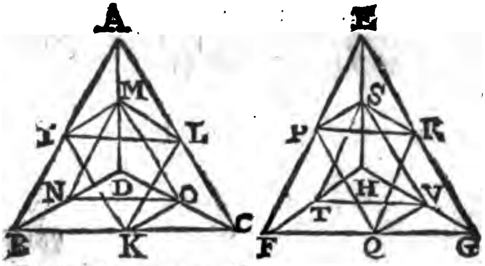


Every Pyramide ABDC having a triangular baſe, may be divided into two Pyramides AEGH, HIKC, equal, and like one to the other, having baſes triangular, and like to the whole ABDC; and into two equal Priſms, BFGEIH, EGDHIK; which two Priſms are greater than the half of the whole Pyramide ABDC.

Divide the ſides of the pyramide into two parts at the points E, F, G, H, I, K, and join the right lines EF, FG, GE, EI, IF, FK, KG, GH, HE. Becauſe the ſides of the pyramide are proportionally

cut, *a* thence HI,AB; and GF, AB; and IF,DC; a 2. 6.
and HG, DC,*&c.* are parallels, and conſequently
HI.FG; and GH,FI are alſo parallels. therefore
it is apparent that the triangles ABD,AEG,EBF,
FDG, HIK, *b* are equiangular, and that the four b 29. 1.
laſt are *c* equal: in like manner the triangles c 26. 1.
ACB,AHE,EIB,HIC,FGK are equiangular; and
the four laſt are equal one to the other. Alſo the
triangles BFI, FDK, IKC, EGH; and laſtly, the
triangles AHG, GDK, HKC, EFI are like and
equal. Moreover the triangles HIK to ADB,and
EGH to BDC, and EFI to ADC, and FGK to
ABC, *d* are parallel. From whence it evidently d 15. 11.
follows, firſt that the pyramides AEGH, HIKC
are equal, and *e* like to the whole ABDC,and to e 10. *def.*
one another. Next, that the ſolids BFGEIH, 11.
FGDIHK, are priſms, and that of equal heighth,
as being placed between the parallel planes ABD,
HIK, but the baſe BFGE is *f* double of the baſe f 2. *ax.* 1.
FDG, wherefore the ſaid priſms are equal; g 40. 11.
whereof the one BFGEIH is greater than the
pyramide BFEI, that is, then AEGH, the whole
than its part; and conſequently the two priſms
are greater than the two pyramides, and ſo exceed the half of the whole pyramide ABDC,
Which was to be dem.

PROP. IV.

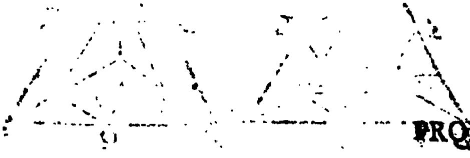


If there be two pyramides ABCD, EFGH, of the ſame altitude, having triangular baſes ABC, EFG;

and either of them be divided into two pyramides (AILM, MNOD; and EPRS, STVH) equal one to the other, and like to the whole, and into two equal priſms (IBKLMN, KLCNMO; and PFQRST, QRGTSV;) and if in like manner either of thoſe pyramides made by the former diviſion be divided, and this be done continually; then as the baſe of one pyramide is to the baſe of the other pyramide, ſo are all the priſms which are in one pyramide, to all the priſms which are in the other pyramide, being equal in multitude.

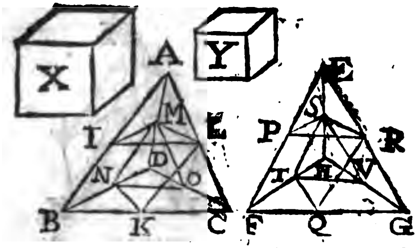
a 15. 5. For (applying the conſtruction of the prece-
b 22. 6. dent Prop.) BC.KC *a* :: FG.QG. *b* therefore the
c 2. 6. &c. triangle ABC is to the like triangle LKC as
d 16. 5. EFG is to *c* the like RQG. therefore by permu-
e ſch. 34. tation ABC.EFG *d* :: LKC. RQG *e* :: the priſm
11. KLCNMO. QRGTSV (for theſe are of equal
f 7. 5. altitude) *f* :: IBKLMN. PFQRST. *g* wherefore
g 12. 5. the triangle ABC. EFG :: the priſm KLCMNO
+IBKLMN. the priſm QRGTSV + PFQRST. *Which was to be dem.*

But if the pyramides MNOD, AILM; and EPRS, STVH, be further divided, in like manner the four new priſms made hereby ſhall be to the four produced before, as the baſes MNO and AIL are to the baſes STV, and EPR; that is, as LKC to RQG, or as ABC to EFG. *h* wherefore all the priſms of the pyramide ABCD are to all the priſms of the pyramide EFGH as the baſe ABC is to the baſe EFG.



PROP.

PROP. V.



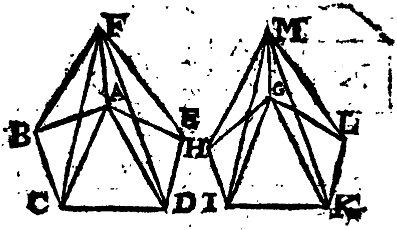
Pyramides ABCD, EFGH, being under the ſame altitude, having triangular baſes ABC, EFG, are one to another as their baſes ABC, EFG, are.

Let the triangles ABC. EFG :: ABCD. X. I ſay X is equal to the pyramide EFGH. For if it be poſſible, let X be ⊐ EFGH. and let the exceſs be Y, divide the pyramide EFGH into priſms and pyramides. and the other pyramides in like manner, *a* till the pyramides left EPRS, SVTH, be leſs than the ſolid Y. Therefore ſince a 1. 10.
the pyramide EFGH = X + Y, it is manifeſt that the remaining priſms PFQRST, QRGTSV are greater than the ſolid X. Conceive the pyramide ABCD divided after the ſame manner; *b* b 4. 11.
then will be the priſm IBKLMN + KLCNMO. PFQRST + QRGTSV :: ABC. EFG *c* :: the c *hyp.*
pyr. ABCD. X. *d* therefore X ⊏ the priſm PFQRST + QRGTSV; which is contrary to that d 14. 5.
which was affirmed before.

Again, conceive X ⊏ the pyr. EFGH. and make the pyr. EFGH. Y :: X the pyr. ABCD *e* e *hyp.* and
:: EFG. ABC. Becauſe EFGH *f* ⊐ X, *g* thence Y *cor.* 4. 5.
⊐ the pyr. ABCD. which is ſhewn before to be f *ſuppoſ.*
impoſſible. Therefore I conclude, that X is equal g 14. 5.
to the pyr. EFGH. *Which was to be dem.*

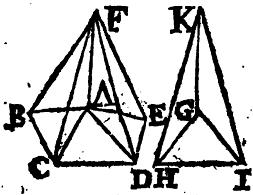
PROP.

PROP. VI.



Pyramides ABCDEF, GHIKLM, conſiſting under the ſame altitude, and having polygonous baſes ABCDE, GHIKL, are to one another as their baſes ABCDE, FGHIKL are.

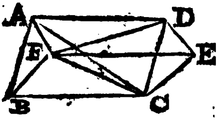
Draw the tight lines AC,AD,GI,GK. then is
a 5. 12. the baſe ABC.ACD *a* :: the pyr. ABCF.ACDF.
b 18. 5. *b* therefore by compoſition ABCD. ACD :: the
pyr. ABCDF. ACDF. *a* but alſo ACD. ADE ::
c 22. 5. the pyr. ACDF. ADEF. *c* therefore of equality
ABCD. ADE :: ABCDF. ADEF. and *b* thence
by compoſition ABCDE. ADE :: the pyr. ABCDEF. ADEF. moreover ADE. GKL *d* :: the pyr.
d 5. 12. ADEF. GKLM ; and as before, and inverſely
GKL. GHIKL :: the pyr. GKLM. GHIKLM.
c therefore again of equality ABCDE.GHIKL::the
pyr. ABCDEF. GHIKLM. *Which was to be dem.*



If the baſes have not ſides of equal multitude, the demonſtration will proceed thus.
e 5. 12. The baſe ABC. GHI *e*
:: the pyr. ABCF. GHIK. *e* and ACD. GHI
:: the pyr. ACDF. GHIK. *f* therefore the baſe ABCD. GHI :: the pyr.
f 4. 5. ABCDF.GHIK. *e* Moreover the baſe ADE.GHI
:: the pyr. ADEF. GHIK. *f* therefore the baſe
ABCDE. GHI :: the pyr. ABCDEF. GHIK.

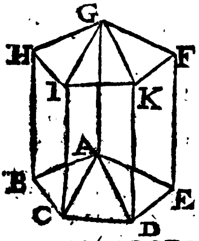
PROP.

PROP. VII.



Every Priſm, ABCDEF, having a triangular baſe, may be divided into three Pyramides ACBF, ACDF, CDFE, equal one to the other, and having triangular baſes.

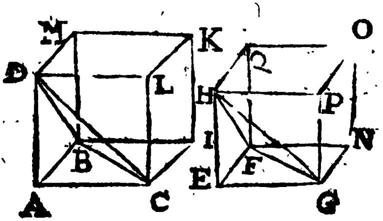
Draw the diameters of the parallelograms,
AC, CF, FD. Then the triangle ACB is *a* = a 34. 1.
ACD. *b* therefore the pyramides of equal heighth b 5. 11.
ACBF. ACDF. are equal. In like manner the
pyr. DFAC = the pyr. DFEC. but ACDF and
DFAC are one and the ſame pyramide. *c* there- c 1. ax. 1.
fore the three pyramides ACBF, ACDF, DFEC, into which the Priſm is divided, are equal one to the other. *Which was to be dem.*



Hence, every pyramide is the third part of the Priſm that has the ſame baſe and heighth with it; or every priſm is treble of the pyramide that has the ſame baſe and heighth with it.

For reſolve the polygonous priſm ABCDEGHIKF into triangular Priſms; and the
pyramide ABCDEH into triangular pyramides;
a then all the parts of the priſm ſhall be treble a 7. 12.
to all the parts of the pyramide, *b* conſequent- b 1. 5.
ly the whole priſm ABCDEGHIKF is treble to the whole pyramide ABCDEH. *Which was to be dem.*

PROP. VIII.



Like Pyramides ABCD, EFGH, which have triangular baſes ABC, EFG are in triple proportion of that in which their ſides of like proportion AC, EG, are.

a 27. 11. *a* Compleat the parallelepipedons ABICDM-
b 9.*def*.11. KL, EFNGHQOP, which *b* are like, and *c*
c 28. 11. ſextuple of the pyramides ABCD, EFGH. *d* and
and 7. 12. therefore in the ſame proportion with them one
d 15. 5. to another, *e* that is, triple of that of the ſides
e 33. 11. of like proportion, *&c.*

Coroll.

Hence, alſo like polygonous pyramides have proportion tripled to that of the ſides of like proportion; as may eaſily be proved by reſolving the ſame into triangular pyramides.

PROP. IX.

See the preceding Scheme.

In equal pyramides ABCD, EFGH, having triangular baſes ABC, EFG, the baſes and altitudes are reciprocal; And pyramides having triangular baſes, whoſe altitudes and baſes are reciprocal, are equal.

1. *Hyp.* The compleated parallelepipedons ABIC-
a 28. 11. DMKL, EFNGHQOP are *a* ſextuple of the equal
and 7. 12. pyramides ABCD, EFGH (either of either) and ſo
equal

equal one to the other. therefore the altitude
(H.) the altitude (D) *b* :: ABIC. EFNG *c* :: b 34. 11.
ABC. EFG. *Which was to be dem.* c 15. 5.

2. *Hyp.* The altitude (H.) the altitude (D) *d* :: d *hyp.*
ABC. EFG *e* :: ABIC. EFNG. *f* therefore the e 15. 5.
parallelepipedons ABICDMKL, EFNGHQOP f 34. 11.
are equal. *g* conſequently alſo the pyramides g 6. *ax.*1.
ABCD, EFGH being ſubſextuple of the ſame,
are equal. *Which was to be dem.*

The ſame is applicable to polygonous pyramides; for they may alſo in like manner be reduced to triangulars.

Coroll.

Whatſoever is demonſtrated of pyramides in Prop.6. 8,9. does likewiſe agree to any ſort of priſms; ſeeing they are triple of the pyramides that have the ſame baſe and altitude with them. Therefore

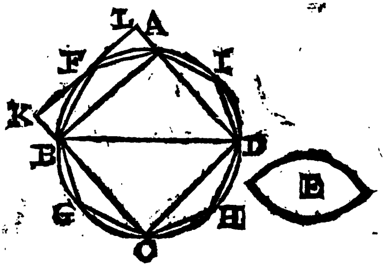
1. The proportion of priſms of equal altitude is the ſame with that of their baſes.
2. The proportion of like priſms is triple of that of the ſides of like proportion.
3. Equal priſms have their baſes and altitudes reciprocal; and priſms which are ſo reciprocal, are equal.

Schol.

From what is hitherto demonſtrated the dimenſion of any priſms and pyramides may be collected.

a The ſolidity of a priſm is produced of the a 1.*cor.*12.
altitude multiplied into the baſe; *b* and there- & *ſch.*40.
fore likewiſe that of a pyramide, of the third 11.
part of the altitude multiplied into the baſe. b 7. 12.

PROP. X.



Every Cone is the third part of a cylinder having the ſame baſe with it ABCD, and the altitude equal.

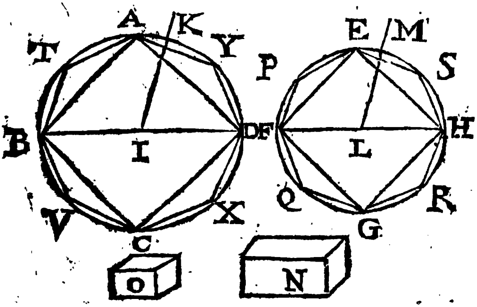
If you deny it, then firſt let ſuch cylinder be more than triple to the cone, and let the exceſs be E. A priſm deſcribed on a ſquare in the circle ABCD *a* is ſubduple of a priſm deſcribed upon a ſquare about the circle, being equal to it and the cylinder in the heighth. Therefore a priſm upon the ſquare ABCD exceeds the half of the cyl. and likewiſe a priſm upon the baſe AFB, of equal heighth to the cylinder, *b* is greater than the half of the ſegment of the cylinder AFB, continue an equal biſection of the arches, and ſubſtract the priſms till the remaining ſegments of the cylinder, namely at AF,FB,*&c.* become leſs than the ſolid E. Therefore the cyl. — ſegm. AF,FB,*&c.* (the priſm on the baſe AFBGCHDI) *c* is greater than the cylinder — E (*d* the triple of the cone.) therefore the pyramide, *e* a third part of the ſaid priſm (being placed on the ſame baſe, and of the ſame heighth) is greater than the cone of equal heighth on the baſe ABCD a circle *i. e.* the part greater than the whole. *Which is abſurd.*

See the ſecond figure of this Book.
a *ſch.* 7. 4. *and cor.* 9. 12.
b *ſch.*27.3. *and cor.* 9, 12.
c 5. *ax.* 1.
d *hyp.*
e *cor.* 7.12.

But if the cone be affirmed to be greater than the third part of the cylinder, then let the exceſs be E. Detract the pyramides from the cone, as you did in the firſt part the priſms from the cylin-

der, till ſome ſegments of the cone remain, conceive at AF, FB, EG, *&c.* leſs than the ſolid E. therefore the cone — E (*f* ⅓ of the cylinder) ⊐ f *hyp.*
the pyr. AFBGCHDI (the cone — ſegm. AF,FB, *&c.*) therefore the priſm triple to the pyramide (*viz.* of equal heighth, and on the ſame baſe) is greater than the cylinder, on the baſe ABCD, the part than the whole, *Which is abſurd.* Wherefore it muſt be granted, that the cylinder is equal to triple of the cone.

PROP. XL.



Cylinders and Cones ABCDK, EFGHM, being under the ſame altitude, are to one another as their baſes ABCD, EFGH.

Let the circle ABCD. the cir. EFGH ∷ the cone ABCDK.N. I ſay N is equal to the cone EFGHM.

For if it be poſſible, let N be ⊐ the cone EFGHM, and let the exceſs be O. The preparation and argumentation of the prec. Prop. being ſuppoſed; then ſhall O be greater than the ſegments of the cone EP,PF,FQ,*&c.* and ſo the ſolid N ⊐ the pyr. EPFQGRHSM. in the circle ABCD *a* a 3 o.3.*and*
make a like polygonous figure ∽ TBVCXDY. Be 1. *poſt.*
cauſe the pyr. ABVYK. the pyr. EFQSM *b* ∷ b 6. 12.
ATBVY. the polyg. EPFQS *c* ∷ the cir. ABCD. c *cor.*2.22.
the cir. EFGH *d* ∷ the cone ABCDK.N. *e* thence d *hyp.*
the pyr. EPFQGRHSM ſhall be ⊐ N. contrary e 14. 5.
to what was affirmed before. Again conceive N

the cone EFGHM. and make the cone EFGH-
f *hyp. and* MO :: N. the cone ABCDK *f* :: the circle EF-
by inver- GH. ABCD. *g* therefore O ⊐ the cone ABCDK;
ſion. *which is abſurd*, as appears by what is ſhewn in
g 14. 5. the firſt part.

Therefore rather admit ABCD. EFGH :: the cone ABCDK. EFGHM. *Which was to be dem.*

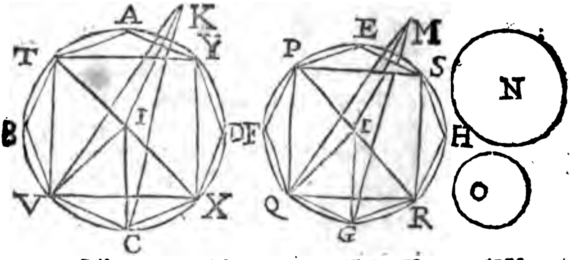
The ſame may be demonſtrated of cylinders, if cylinders and priſms be conceived in the place of cones and pyramides. therefore, *&c.*

Schol.

Hence is gathered the dimenſion of all ſorts of cylinders and cones. The ſolidity of a right cylin-
a 1. *Prop.* der is produced of the circular baſe (*a* the di-
de dimenſ. menſion whereof is to be learnt out of *Archime-*
circ. *des*) multiplied into the heighth; *b* whence
b 11. 12. in like manner that of every cylinder.

Therefore the ſolidity of a cone is produced of the third part of the altitude multiplied into the baſe.

PROP. XII.



Like cones and cylinders ABCDK, EFGHM, are in triplicate proportion of that of the diameters TX, PR, of their baſes ABCD, EFGH.

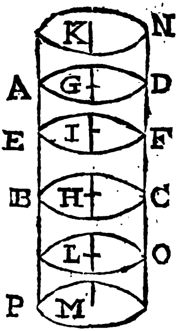
Let the cone A have to N triplicate proportion of TX to PR. I ſay N is = the cone EFGHM. For if it be poſſible, let N be ⊐ EFGHM. and let the exceſs be O. therefore, N ⊐ the pyr. EPFQGRHSM. Let the axes of the cones be IK, LM, and join the right lines VK, CK, VI, CI, and QM,

GM, QL, GL. Becauſe the cones are like, *a* a 24. *def.*
thence VI. IK :: QL. LM. but the angles VIK, 11.
QLM *b* are right angles. *c* therefore the triangles b 18. *def.*
VIK,QLM are equiangular. *d* whence VC, 11.
VI :: QG. QL. alſo VI. VK :: QL.QM. therefore c 6. 6.
of equality VC. VK :: QG.QM. *e* moreover d 4. 6.
VK.GK :: QM. MG. therefore again of equality e 7. 5.
VC. CK :: QG. GM. *f* therefore the triangle f 5. 6.
VKC, QMG are like : and by the ſame reaſon
the other triangles of this pyramide are like
to the other of that ; *g* wherefore the pyramides g 9.*def.*11
themſelves are like. *h* But they are in triplicate h *cor.*8.12.
proportion of that of VC to QG. *k* that is, of k 4. 6.
VI to RL, *l* or TX to PR. *m* therefore the pyr. l 15. 5.
AIBVCXDYK. the pyr. EPFQGRHSM :: the m *hyp. and*
cone of ABCDK. N. *n* whence the pyr. EPFQGRHSM 11. 5.
⊐ N. which is repugnant to what n 14. 5.
was affirmed before.

Again, take N ⊏ the cone EFGHM. make
the cone EFGHM. O :: N. the cone ABCDK
o :: the pyr. EPRM. ATCK *p* :: GQ. VC thrice o *before,*
:: *q* PR. TX thrice. but O *r* is ⊐ ABCDK. *and in-*
which was before ſhewn to be repugnant. *verſely.*
Wherefore N = the cone EFGHM. *Which* p *cor.*8.12.
was to be dem. q 4. 6.

But foraſmuch as what proportion ſoever r 14. 5.
cones have, alſo cylinders being triple of them,
have the ſame ; therefore cylinder to cylinder
ſhall have proportion triplicate of the diameters
of the baſes.

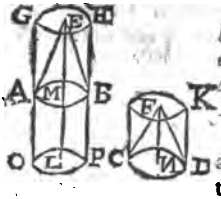
PROP. XIII.



If a cylinder ABCD, be divided by a plane EF parallel to the oppoſite planes BC,AD, then as one cylinder AFED is to the other cylinder EBCF, ſo is the axis GI to the axis HI.

The axis being produced,
a 3. 1. *a* take GK=GI, and HL=
IH=LM, and conceive planes
drawn at the points K, L,M,
parallel to the circles AD,
b 11. 12. BC, *b* therefore the cylinder
FD = the cyl. AN. and the
cyl. EC *b* = BO *b* = OP. therefore the cylinder EN is as multiple of the cylinder ED as the axis IK is of the axis IG. and in like manner the cylinder FP is as multiple of the cylinder
BF, as the axis IM is of the axis IH. but as
c 11. 12. IK is =,⊏,⊐ IM, *c* ſo is the cylinder EN =,⊏,
d 6. *def.* 5. ⊐ EP. *d* therefore the cylinder AEFD. the cyl.
EBCF :: GI. IH. *Which was to be dem.*

PROP. XIV.



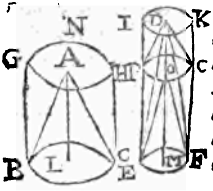
Cones AEB, CFD, and cylinders AH,CK conſiſting upon equal baſes AB, CD, are to one another as their altitudes ME, NF.

The cylinder AH, and the axis EM being produced,
take ML=FN; and at the
a 11. 12. point L draw a plane parallel to the baſe AB, *a*
b 13. 12. then ſhall the cyl. AP be = CK. *b* but the cyl. AH.
AP(CK) :: ME.ML (NP.) *Which was to be dem.*

* *apply* 9. The ſame may be affirmed of cones ſubtriple
and 7. 12. of cylinders; * as alſo of priſms and pyramides.

PROP.

PROP. XV.



In equal cones BAC, EDF, and cylinders BH, EK, the baſes and altitudes are reciprocal BC. EF :: MD. LA.) And cones and cylinders, whoſe baſes and altitudes are reciprocal, are equal one to another.

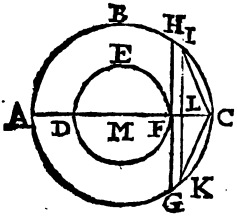
If the altitudes be equal then the baſes are equal too, and the thing is evident. If unequal, then take away MO = LA.

1. *Hyp.* Then is MD. MO (*a* LA) *b* :: the a 14. 12.
cyl. EK. (*c* BH) EQ *d* :: the cir. BC. EF. *Which* b *conſtr.*
was to be dem. c *hyp.*

2. *Hyp.* BC. EF *e* :: DM. OM (LA) *f* :: the d 11. 12.
cyl. EK. FQ *g* :: BC. EF *h* :: BH. EQ. Therefore e *hyp.*
the cylinder EK = BH. *Which was to be dem.* f 11. 12.

The ſame argument may be uſed for cones. g 11. 5.
h 11. 12.

PROP. XVI.
k 9. 5.

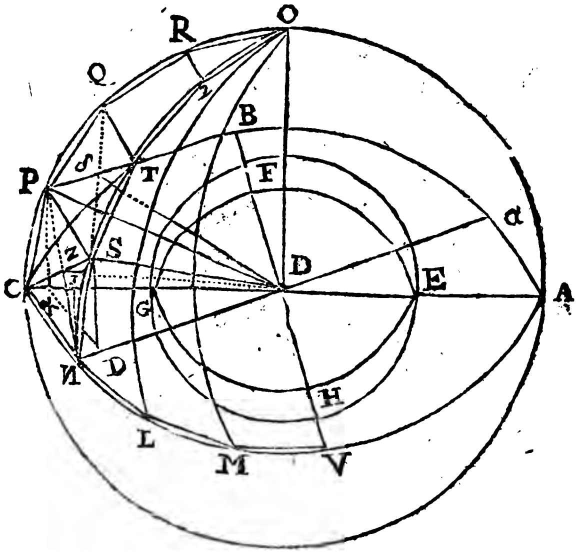


Two unequal circles ABCG, DEF, having the ſame center M, to inſcribe in the greater circle ABCG a polygonous figure of equal, and even ſides, which ſhall not touch the leſſer circle DEF.

Through the center
M draw the line AC cutting the circle DEF in
F, from whence raiſe a perpendicular FH. *a* di- a 30. 3.
vide the ſemicircle ABC into two equal parts;
and the half thereof BC alſo; and ſo do conti-
nually. *b* till the arch IC become leſs than the b 1. 10.
arch HC, from I let fall the perpendicular IL. It
is manifeſt that the arch IC meaſures the whole
circle, and that the number of arches is even, and

c *ſch.*16.4. ſo that the ſubtended line IC is the ſide of the
polygon that may be inſcribed without touch-
d *cor.*16.3. ing the leſſer circle DEF. For HG *d* touches
e 28. 1. the circle DEF. *e* to which IK is parallel, and
f 34.*def.*1. placed outwardly ; *f* wherefore IK does not
touch the circle DEF ; much leſs do CI, CK,
and the other ſides of the polygonon more re-
mote from the center. *Which was to be done.*

PROP. XVII.



Two ſpheres ABCV, EFGH, conſiſting about the ſame center D, being given, to inſcribe a ſolid of many ſides (or Polyedron) in the greater ſphere ABCV, which ſhall not touch the ſuperficies of the leſſer ſphere EFGH.

Let both the ſpheres be cut by a plane paſſing by the center making the circles EFGH, ABCV; and the diameters AC, BV drawn, cutting perpendicularly. In the circle ABCV, *b* inſcribe the a 16. 12.
equilateral polygone VMLNC, *&c.* not touching the circle EFGH: then draw the diameter Nα, and erect DO perpendicular to the plane ABC. by DO, and by the diameters AC, Nα, conceive planes DOC, DON erected, which ſhall be *b* perpendicular to the circle ABCV, and ſo in b 18. 11.
the ſuperficies of the ſphere make *c* the quadrants c *cor.* 33. 6.
DOC, DON. In which let the right lines CP, PQ, QR, RO, NS, ST. Tγ, γO *d* be fitted, equal, d 4. 1.
and of equal multitude with CN, NL, *&c.* make the ſame conſtruction in the other quadrants OL, OM, *&c.* and in the whole ſphere. Then I ſay the thing required is done.

From the points P, S, to the Plane ABCV draw the perpendiculars PX, SY, *e* which ſhall fall on e 38. 11.
the ſections ACNα. Therefore becauſe both *f* the f 12. *ax.*
right angles PXC, SYN, *g* and PCX, SNY inſiſt- g 27. 3.
ing on *h* equal circumferences, *f* are equal, the h 32. 1.
triangles alſo PCX, SNY *b* are equiangular. Where- k *conſtr.*
fore being PC *k* = SN, *l* alſo is PX = SY, *l* and l 26. 1.
XC = YN; *m* whence DX = DY. *n* and therefore m 3. *ax.* 1.
DX. XC :: DY. YN. *o* therefore YX, NC are pa- n 7. 5.
rallels. but becauſe PX, SY are equal, and ſince o 2. 6.
being perpendicular to the ſame plane ABCV, p 6. 11.
they are alſo *p* parallels, *q* therefore YX, SP ſhall q 33. 1.
be equal and parallel. *r* whence SP, NC, are pa- r 9. 12.
rallel one to the other; and ſo the *ſ* quadrilate- ſ 7. 11.
ral NCPS. and by the ſame reaſon SPQT, t 2. 11.
TQRG, as alſo the *t* triangle γ RO are ſo many planes. In like manner the whole ſphere may be ſhewn full of ſuch quadrilaterals and triangles. wherefore the figure inſcribed is a polyedron.

From the center D *u* draw DZ perpendicular to u 11. 11.
the plane NCPS; and join ZN, ZC, ZS, ZP. Be- x 4. 6.
cauſe DN. NC *x* :: DY. YX. thence NC is *y* ⊏ YX. y 14. 5.

(SP.) and likewiſe SP ⊏ TQ, and TQ ⊏ γ R.
And becauſe the angles DZC, DZN, DZS, DZP
z 3. *def.* 11. *z* are right, and the ſides DC, DN, DS, DP, *a* equal,
a 15. *def.* and DZ common, *b* thence ZC, ZN, ZS, ZP are e-
11. qual one to the other; and conſequently about
b 47. 1. the quadrilateral NCPS *c* a circle may be deſcri-
c 15. *def.* 1. bed, in which (becauſe NS, NC, CP, are *d* equal,
d *conſtr.* and NC ⊏ SP) NC *e* ſubtends more than the qua-
e 28. 3. drant, *f* therefore the ang. NZC at the center is
f 33. 9. obtuſe, *g* therefore NCq ⊏ 2 ZCq (ZCq+ZNq.)
g 12. 2. Let NI be drawn perpendicular to AC. there-
h 32. 1. fore ſince the angle ADN (*h* DNC+DCN) *k* is
k 9. *ax.* 1. obtuſe, the half of it DCN ſhall be greater than
l 5. 1. the half of a right angle; and ſo that which re-
mains of the right ang. CNI ſhall be leſs than it
n 19. 1. *n* whence IN ⊏ IC. therefore NCq (NIq + ICq)
o 47. 1. *o* ⊐ INq. therefore IN ⊏ ZC. and conſequently
p 47. 1. DZ *p* ⊏ DI. but the point I is *q* without the ſphere
q *cor.* 19. EFGH. and ſo much more the point Z. wherefore
12. the plane NCPS, (whoſe *r* next point to the cen-
r 47. 1. ter is Z) does not touch the ſphere EFGH. And
if a perpendicular Dδ be drawn to the plane
SPQT, the point δ, and ſo alſo the plane SPQT
is yet further removed from the center, which
is alſo true of the other planes of the polyedron.
Therefore the polyedron ORPQCN, &c. in-
ſcribed in the greater ſphere, does not touch the
leſſer. *Which was to be done.*

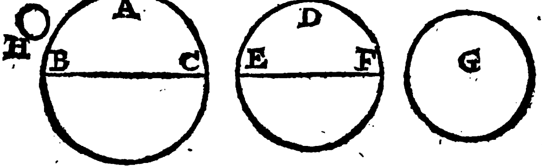
Coroll.

Hence it follows, *That if in any other ſphere a ſolid polyedron, like to the aboveſaid ſolid polyedron, be inſcribed, the proportion of the polyedron in one ſphere to the polyedron in the other is triplicate of that of the diameters of the ſpheres.*

For if right lines be drawn from the centers of the ſpheres to all the angles of the baſes of the ſaid polyedrons, then the polyedrons will be divided into pyramides equal in number and like;

whoſe homologous ſides are ſemidiameters of the ſpheres ; as appears, if the leſſer of theſe ſpheres be conceived deſcribed within the greater about the ſame center. For the right lines drawn from the center of the ſphere to the angles of the baſes will agree one to the other by reaſon of the likeneſs of the baſes; and ſo will like pyramides be made. Wherefore ſince every pyramide in one ſphere to every pyramide like it in the other ſphere, *a* has proportion triple to a cor. 8. 12. that of the homologous ſides, that is, of the ſemidiameters of the ſpheres ; and *b* as one py- b 12. 5. ramide is to one pyramide, ſo all the pyramides, that is, the ſolid polyedron compoſed of theſe, are to all the pyramides, that is, the ſolid polyedron compoſed of the others ; therefore the polyedron of one ſpace ſhall have to the polyedron of the other ſphere, proportion triple of that of the ſemidiameters, *c* and ſo of the dia- c 15. 5. meters of the ſpheres.

PROP. XVIII.



Spheres BAC, EDF, are in triplicate proportion one to the other of that in which their diameters BC, EF.

Let the ſphere BAC be to the ſphere G in triple proportion of that of the diameter BC to the diameter EF. I ſay G = DEF. For if it be poſſible, let G be $\sqsupset$ EDF. and conceive the ſphere G concentrical with EDF. In the ſphere of EDF *a* inſcribe a polyedron not touching the a 17. 12. ſphere G, and a like polyedron in the ſphere BAC, b *cor.* 17. 12. BAC. Theſe polyedrons *b* are in triplicate proportion of the diameters BC, EF. *c* that is, of the c *hyp.* ſphere BAC to G. *d* Conſequently the ſphere d 14. 5. G is greater than the polyedron inſcribed in the ſphere EDF, the part than the whole.

Again, if it be poſſible, let the ſphere G be ⊏ EDF. and as the ſphere EDF is to another e *hyp. in-verſ.* ſphere H, ſo let G be to BAC, *e* that is, in proportion of the diameter EF to BC. there- f 14. 5. fore ſince BAC *f* ⊏ H, we ſhall incur the abſurdity of the firſt part. wherefore rather the ſphere G = EDF. *Which was to be dem.*

Coroll.

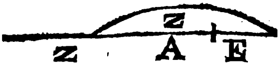
Hence, As one ſphere is to another ſphere, ſo is a Polyedron deſcribed in that, to a like polyedron deſcribed in this.

The End of the twelfth Book.

THE THIRTEENTH BOOK OF EUCLIDE's ELEMENTS.

PROP. I.

If a right line z *be divided according to extreme and mean proportion* (z. a :: a. e.) *the ſquare of the half of the whole line* z, *and of the greater ſegment* a, *as one line is quintuple to that which is deſcribed of half of that whole line z.*



I ſay Q. $a + \frac{1}{2}z = 5$ Q: $\frac{1}{2}$ z. *a* that is, $aa + \frac{1}{4}zz + za$ $= zz + \frac{1}{4}zz$. *b* or $aa + za = zz$. For $ze + za$ *c* $= zz$. and ze *d* $= aa$. *e* therefore $aa + za = zz$. *Which was to be dem.*

a 4. 2.
b 3. *ax.* 1.
c 2. 2.
d *hyp.* and 16. 6.
e 2. *ax.* and 1. *ax.*

PROP. II.

See the 1. *Scheme.*

If a right line $\frac{1}{2}z + a$ *be in power quintuple to a ſegment of it ſelf* $\frac{1}{2}$ z. *the line double of the ſaid ſegment* (z) *being divided according to extreme and mean proportion, the greater ſegment is* (a) *the other part of the right line at firſt given* $\frac{1}{2}z + a$.

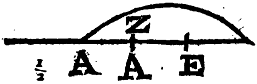
I ſay z. a :: a. e. For becauſe by the hyp. * $aa + \frac{1}{4}zz + za = zz + \frac{1}{4}zz$; or $aa + za = zz$ *a* $= ze + za$, *b* thence ſhall aa be $= ze$. *c* wherefore z. a :: a. e. *Which was to be dem.*

* 4. 2.
a 2. 2.
b 3. *ax.* 1.
c 17. 6.

PROP.

PROP. III.

If a right line z *be divided according to extreme and mean proportion* (z. a :: a . e.) *the line made of the leſs ſegment* e *and half of the greater ſegment* a, *is in power quintuple to the ſquare, which is deſcribed of the half line of the greater ſegment* a.

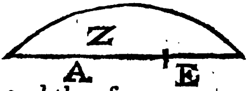


I ſay $Q{:}\ e + \frac{1}{2}a = 5\ Q{:}\ \frac{1}{2}a$. *a* that is $ee + \frac{1}{4}aa + ea = aa + \frac{1}{4}aa$. *b* or $ee + ea = aa$. For $ee + ea$ *c* $= ze$ *d* $= aa$. *Which was to be dem.*

a 4. 2.
b 3. *ax.*
c 3. 2.
d *hyp.* and 17. 6.

PROP. IV.

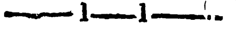
If a right line z *be cut according to extreme and mean proportion* (z. a :: a. e.) *the ſquare made of the whole line* z, *and that made of the leſſer ſegment* e, *both together, are triple of the ſquare made of the greater ſegment* a.



I ſay $zz + ee = 3\ aa$. *a* or $aa + ee + 2\ ae + ee = 3\ aa$. For $ae + ee$ *b* $= ze$ *c* $= aa$. *d* therefore $aa + 2\ ae + 2\ ee = 3\ aa$. *Which was to be dem.*

a 4. 1.
b 3. z.
c 17. 6.
d 2. *ax.*

PROP. V.



If a right line AB be cut according to extreme and mean proportion in C, and a line AD, equal to the greater ſegment BC, added to it, the whole right line DB is divided according to extreme and mean proportion; and the greater ſegment the right line AB given at the beginning.

a *hyp.* For becauſe AB. AD *a* :: AC.CB. and by inverſion AD. AB :: CB. AC. therefore by compoſition DB. AB :: AB. AC (AD.) *Which was to be dem.*

Schol.

But if BD. BA :: BA. AD. then ſhall be BA. AD :: AD. BA — AD. For by diviſion is BD — BA (AD.) BA :: BA — AD. AD. therefore inverſely BA. AD :: AD. BA — AD.

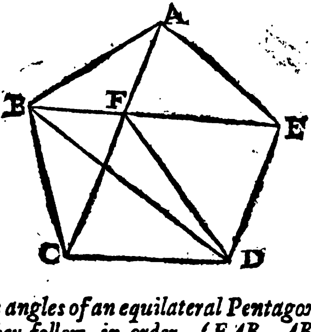
PROP. VI.

D A C B

If a rational right line AB be cut according to extreme and mean proportion in C, either of the ſegments (AC, CB) is an irrational line of that kind which is called apotome or reſidual.

To the greater ſegment AC *a* add AD = $\frac{1}{2}$ AB. a 3. 1.
b therefore DCq = 5 DAq. *c* therefore DCq $\sqcap\!\!\sqcup$ b 1. 13.
DAq. conſequently *d* ſince AB, *e* and ſo the c 6. 10.
half thereof DA are $\acute{\rho}$, likewiſe DC is $\acute{\rho}$. But d *hyp.*
becauſe 5. 1 :: not Q. Q. *f* thence is DC $\not{\sqcap\!\!\sqcup}$ DA. e *ſch.* 12.
g therefore DC — AD, that is, AC, is a reſi- 10.
dual line. Further, becauſe ACq *h* = AB × f 9. 10.
BC, and AB is $\acute{\rho}$. *k* likewiſe EC is a reſidual g 74. 10.
line. *Which was to be dem.* h 17. 6.
k 98. 10.

PROP. VII.



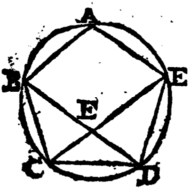
If three angles of an equilateral Pentagone ABCDE whether they follow in order, (EAB, ABC, BCD,) or not, (EAB, BCD, CDE) be equal, the Pentagone ABCDE ſhall be equiangular.

Let the right lines BE, AC, BD, be ſubtended to the equal angles in order.

Being the ſides EA, AB, BC, CD, and the in-
a *hyp.* cluded angles *a* are equal, therefore ſhall the ba-
b 4. 1. ſes BE, AC, BD, *c* and the angles AEB, ABE,
c 4. *and* 5. BAC, BCA, be equal. *d* wherefore BF = FA. *e*
1. and conſequently FC = FE; therefore the trian-
d 6. 1. gles FCD, FED, are equilateral one to the other:
e 3. *ax.* 1. *f* whence the angle FCD = FED. *g* conſequently
f 8. 1. the angle AED = BCD. In like manner the ang.
g 2. *ax.* 1. CDE is equal to the reſt; wherefore the pentagone is equiangular. *Which was to be dem.*

But if the angles EAB, BCD, CDE, which
h 4. 1. are not in order, be ſuppoſed equal. *h* then ſhall
k 5. 1. the angle AEB be = BDC, and BE = BD. *k*
l 2. *ax.* and thence the angle BED = BDE. *l* conſequently the whole angle AED = CDE. therefore becauſe the angles A, E, D, in order, are equal, as before, the pentagone ſhall be equiangular. *Which was to be dem.*

PROP. VIII.

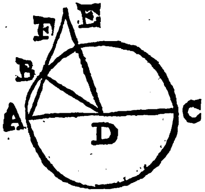


If in an equilateral and equiangular Pentagone ABCDE, two right lines BD, CE, ſubtend two angles BCD, CDE, following in order, thoſe lines do cut one another according to extreme and mean proportion, and their greater ſegments BF or EF are equal to the ſide of the pentagone BC.

a 14. 4. *a* Deſcribe about the pentagone the circle ABD.
b 28. 3. *b* The arch ED is = BC, *c* therefore the angle
c 27. 3. FCD = FDC, *d* therefore the ang. BFC = 2 FCD
d 32. 1. (FCD + FDC. But the arch BAE *b* is = 2 ED,
e 33. 6. and conſequently the angle BCF *e* = 2 FCD =
f 6. 1. BFC. *f* wherefore BF = BC. *Which was to be dem.*
g 27. 3. Moreover becauſe the triangles BCD, FCD, are
h 4. 6. *g* equiangular, *h* therefore BD . DC (BF.) :: CD.

(BF.) FD. and likewiſe EC.EF :: EF. FC. *Which was to be dem.*

PROP. IX.



If the ſide of an Hexagone BE, and the ſide of a Decagone AB, both deſcribed in the ſame circle ABC, be added together, the whole right line AE is cut according to extreme and mean proportion (AE. BE :: BE. AB.) and the greater ſegment thereof is the ſide of the Hexagone BE.

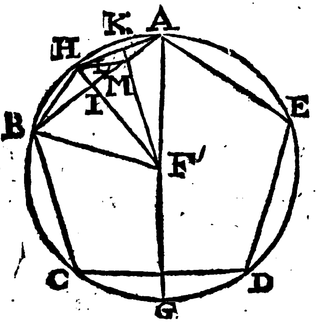
Draw the diameter ADC, and join the right
lines DB, DE. Becauſe the angle BDC *a* = 4 a *hyp. and*
BDA. and the angle BDC *b* = 2 DBA (DAC+ 27. 3.
DBA) thence ſhall DBA (*b* DBE + BED) *c* be b 32. 1.
= 2 BDA *d* = 2 BDE. whence the angle DBA c 7. *ax.* 1.
or DAB *e* = ADE. Therefore the triangles ADE, d 5. 1.
ADB, are equiangular : *f* wherefore AE. AD (*g* e 1. *ax.* 1.
BE) :: AD. (BE.) AB. *Which was to be dem.* f 4. 6.
g *cor.* 15. 4.

Coroll.

Hence, If the ſide of an Hexagone in a circle be cut according to extreme and mean proportion; the greater ſegment thereof ſhall be the ſide of the Decagone in the ſame circle.

PROP.

PROP. X.



If an equilateral Pentagone ABCDE be deſcribed in a circle ABCE, the ſide of the Pentagone AB contains in power both the ſide of a hexagone FB, and the ſide of a decagone AH deſcribed in the ſame circle.

Draw the diameter AG. and biſect equally the arch AH in K. and draw FK, FH, FB, BH, HM.

The ſemicircle AG − the arch AC *a* = AG − AD. that is, the arch CG = GD *b* = AH = HB. therefore the arch BCG = 2 BHK; *c* and ſo the angle BFG = 2 BFK. *d* but the angle BFG = 2 BAG. *e* therefore the angle BFK = BAG. Wherefore the triangles BFM, FAB, *f* are equiangular. *g* whence AB. BF :: BF. KM. *h* therefore AB x BM = BFq. Moreover, the angle AFK *k* = HFK, and FA = FH. *m* wherefore AL = LH. *m* and the angles FLA, FLH are equal, and ſo right angles. therefore the angle LHM *m* = LAM *n* = HBA. therefore the triangles AHB, AMH, *o* are equiangular. wherefore AB. AH :: AH. AM. *q* therefore AB x AM = AHq. So that ſeeing ABq *r* = AB x BM + AB x AM, *ſ* thence ABq = BFq + AHq. *Which was to be dem.*

a 28.3. *and* 3. *ax.*
b *hyp. and* 7. *ax.*
c 33 6.
d 20. 3.
e 1. *ax.* 1.
f 31. 1.
g 4. 6.
k 27. 3.
m 4. 1.
n 27. 3.
o 32. 1.
p 4. 6.
q 17. 6.
r 2. 2.
ſ 2. *ax.*

Coroll.

1. Hence, a right line (FK) which being drawn from the center (F) divides an arch (HA) into two equal ſegments, does alſo divide the right

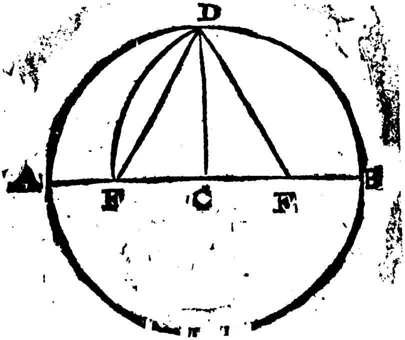
line (HA) ſubtending that arch, perpendicularly into two equal ſegments.

2. The diameter of a circle (AG) drawn from any angle (A) of a pentagone, does divide equally in two both the arch (CD) which the ſide of the pentagone oppoſite to that angle ſubtends, and alſo the oppoſite ſide it ſelf (CD) and that perpendicularly.

Schol.

Here, according to our Promiſe, we ſhall lay down a ready praxis of the 11. *Prop. of the* 4. *Book.*

Problem.

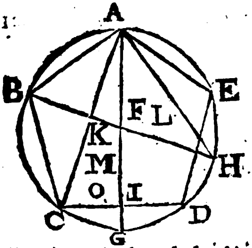


To find out the ſide of a pentagone to be inſcribed in a circle ADB.

Draw the diameter AB, to which erect a perpendicular CD at the center C divide CB equally in E. and make EF = ED. then DF ſhall be the ſide of the pentagone. a 6. 2.

For BF × FC + ECq *a* = EFq *b* = EDq *c* = b *conſtr.*
DCq + ECq. *d* therefore BF × FC + DCq or c 47. 1.
BCq. *e* wherefore BF. BC :: BC. FC. therefore d 3. *ax.*
ſince BC is the ſide of an hexagone, *f* FC ſhall e 7.
be the ſide of a decagone. Conſequently DF f 9. 13.
h $= \sqrt{}$ DCq + FCq *g* is the ſide of a pentagone. g 10. 13.
Which was to be done. h 47. 1.

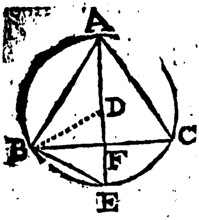
PROP. XI.



If in a circle ABCD, whoſe diameter is rational AG, an equilateral pentagone be inſcribed ABCDE; the ſide of the pentagone AB is an irrational line of that kind which is called a minor line.

Draw the diameter
* 18. 6. BFH, and the right lines AC, AH; and * make
FL $= \frac{1}{4}$ of the radius FH; and CM $= \frac{1}{4}$ CA.
a *cor.* 10. Becauſe the angles AKF, AIC, are *a* right an-
13. gles, and CAI common, the triangles AKF,
b 32. 1. AIC, are *b* equiangular: *c* therefore CI. FK *c* ::
c 4. 6. CA. FA (FB) *d* :: CM. FL. therefore by permu-
d 15. 5. tation FK. FL :: CI. CM *d* :: CD. CK (2 CM)
e 18. 5. and ſo by *e* compoſition CD + CK. CK :: KL.
f 22. 6. FL. *f* conſequently Q: CD + CK (*g* 5 CKq.)
g 1. 13. CKq. :: KLq. FLq. therefore KLq = 5 FLq.
wherefore if BH (ρ) be taken 8, FH ſhall be 4,
FL 1, and FLq 1, BL 5, and BLq 25, KLq 5,
h 9. 10. by which it appears that BL and KL are $\dot{\rho}$ *h* ╤,
k 74. 10. *k* and ſo BK is a reſidual, and KL its congruent
l 9. 10. or adjoining line, but being BLq − KLq = 20,
m *cor.* 8. 6. *l* thence BL ⊤⊥ √ BLq − KLq. *m* whence BK
and 17. 6. ſhall be a fourth reſidual line. Therefore becauſe
n 95. 10. ABq *m* is = HB × BK, *n* ſhall AB be a minor
line. *Which was to be dem.*

PROP. XII.



If in a circle ABEC an equilateral triangle ABC be inſcribed, the ſide of that triangle AB is in power triple to the line AD drawn from D the center of the circle to the circumference.

The diameter being extended to E, draw BE. Be-
cauſe the arch BE *a* = EC, the arch BE is the a *cor.* 10.
ſixth part of the circumference, *b* therefore BE 13.
= DE. hence AEq *c* = 4 DEq (4 BEq) *d* = b *cor.*15.4.
ABq + BEq (+ ADq.) *e* conſequently ABq = c 4. 2.
3 ADq. *Which was to be dem.* d 47. 1,

Corell. e 3. *ax.* 1.

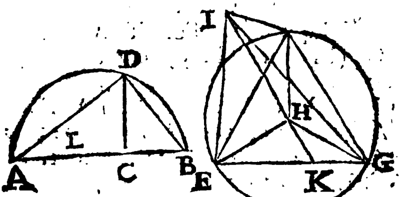
1. AEq. ABq :: 4. 3.

2. ABq. AFq :: 4. 3. *f* For ABq. AFq :: AEq. f *cor.* 8.6.
ABq. *and* 22. 6.

3. DF = FE. For the triangle EBD *g* is e- g *cor.*13.4.
quilateral, *h* and BF perpendicular to ED. *h* h *cor.* 3.3.
therefore EF = FD.

4. Hence, AF = DE + DF = 3 DF.

PROP. XIII.



To deſcribe a pyramide EGFI, and comprehend it in a ſphere given: and to demonſtrate that the diameter of the ſphere AB is in power ſeſquialter of the ſide EF of the pyramide EGFI.

About

a 10. 6. About AB deſcribe the ſemicircle ADB; *a* and
let AC be = 2 CB. from the point C erect the
perpendicular line CD; and join AD, DB, then
at the interval of the radius HE = CD deſcribe
b cor. 15. 4. the circle HEFG, wherein inſcribe the equilateral
c 12. 11. triangle EFG from H *c* erect IH = CA perpendicular
d 3. 1. to the plane EFG. produce IH to K, *d* ſo
that IK = AB; and join the right lines IE, IF, IG.
Then EFGI ſhall be the pyramide required.

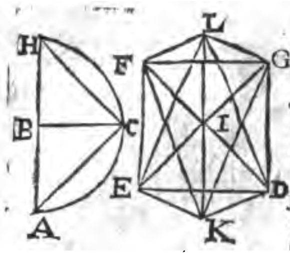
For becauſe the angles ACD, IHE, IHF, IHG,
e conſtr. *e* are right angles; and CD, HE, HF, HG *e* equal.
f 41. 1. *e* and IH = AC; *f* therefore AD, IE, IF, IG
g 20. 6. ſhall be equal among themſelves. But being AC
h 2. *ax.* (2 CB) CB *g* :: ACq. CDq. then ſhall ACq be =
k 12. 13. 2 CDq. therefore ADq *f* = ACq + CDq *h* =
l 1. *ax.* 1. 3 CDq = 3 HEq *k* = EFq. *l* therefore AD, EF,
IE, IF, IG are equal, and ſo the pyramide EFGI
is equilateral. But if the point C be placed
upon H, and AC upon HI, the right lines AB,
m 8. *ax.* IH, *m* ſhall agree, as being equal. Wherefore
the ſemicircle ADB being drawn about the axis
n 15. *def.* 1. AB or IK *n* ſhall paſs by the point E, F, G, *
* 31. *def.* 11. and ſo the pyramide EFGI ſhall be inſcribed in
a ſphere. *Which was to be done.*
o cor. 8. 6. Alſo it is manifeſt that BAq. ADq *o* :: BA.
p conſtr. AC *p* :: 3. 2. *Which was to be dem.*

Coroll.

1. ABq. HEq :: 9. 2. For if ABq be put 9,
then ACq (EFq) ſhall be 9, conſequently HEq
q 12. 13. ſhall be 2.

2. If L be the center, then ſhall AB. LC :: 6. 1.
For if AB be put 6, then AL ſhall be 3. *r* and
thence AC 4. wherefore LC ſhall be 1. Hence
r conſtr. 3. AB. HI :: 6. 4 :: 3. 2. whence
4. ABq. HIq :: 9. 4.

PROP. XIV.



To deſcribe an Octaedron KEFGDL, and comprehend it in the given ſphere, wherein a Pyramide is: and to demonſtrate that AH the diameter of the ſphere is in power double of AC the ſide of that Octaedron.

About H deſcribe the ſemicircle ACH. and
from the center B erect the perpendicular BC.
draw AC, HC, then upon ED=AC. *a* make the a 46. 1.
ſquare EFGD, whoſe diameters DF, EG, cut in
the center I. from I draw IL=AB *b* perpendicu- b 12. 11.
lar to the plane EFGD. produce IL, *c* till IK=IL. c 3. 1.
and join KE,KF,KG,KD,LE,LF,LG,LD; then
ſhall KEFGDL be the Octaedron required.

For AB,BH,FI,IE, *&c.* being ſemidiameters of
equal ſquares are equal one to the other. *d* whence d 4. 1.
the baſes LF, LE, FE, *&c.* of the right-angled
triangles LIE, LIF, FIE, *&c.* are equal, and
conſequently the eight triangles LFE,LEG,LGD,
LDE,KEF,KFG,KGD,KDE, are equilateral, *e* e 27. *def.*
and make an Octaedron, which may be inſcri- 11.
bed in a ſphere, whoſe center is I, and IL or
AB the radius. (becauſe AB, IL, IF, IK, *&c.* *f* f *conſtr.*
are equal.) *Which was to be done.* Moreover, it
is evident that AHq (LKq) *g* = 2 ACq (2 g 47. 1.
LDq.) *Which was to be dem.*

Coroll.

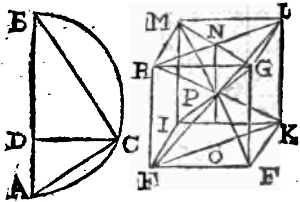
1. Hence it is manifeſt, that in the octaedron the three diameters, EG, FD, LK do cut one the other perpendicularly in the center of the ſphere.

2. Alſo that the three planes EFGD, LEKG, LFKD are ſquares, cutting one another perpendicularly.

3. The Octaedron is divided into two like and equal pyramides EFGDL, and EFGDK, whoſe common baſe is the ſquare EFGD.

15. 11. 4. Laſtly, It follows that the oppoſite baſes of the Octaedron are parallel one to the other.

PROP. XV.



To deſcribe a cube EFGHIKLM, and comprehend it in the ſame ſphere wherein the former figures were; and to demonſtrate that AB the diameter of the ſphere is in power triple to EF the ſide of that cube.

a 10. 6. Upon AB deſcribe a ſemicircle ACB; *a* and make AB = 3 DA, from D raiſe the perpendicular DC, and join BC and AC. Then upon EF = AC
b 46. 11. *b* make the ſquare EFGH, upon whoſe plane let the right lines EI, FK, HM, GL, ſtand perpendicular, being equal to EF, and connect them with the right lines IK, KL, LM, IM: The ſolid EFGHIKLM is a cube, as is ſufficiently apparent from the conſtruction.

In the oppoſite ſquares EFKI, HGLM, draw the diameters EK,FI,HL,MG, by which let the planes EKLH,FI,MG be drawn,cutting one ano-
c *cor.* 39. ther in the line NO. which *c* ſhall divide equally
11. in two parts the diameters of the cube EL,FM,GI,
d 15. *def.* HK, in P the center of the cube, *d* therefore P
11. *and* 14, ſhall be the center of a ſphere paſſing by the an-
def. 11. gular points of the cube. Moreover, ELq *e* = EKq
e 47. 1. + KLq, *e* = 3 KLq, *f* or 3 ACq, but ABq. ACq *g*::
f *conſtr.* BA.DA *f*:: 3.1, *h* therefore AB = EL. wherefore we
g *cor.* 8. 6, have made a cube, *&c.* *Which was to be done.*
h 14. 5.

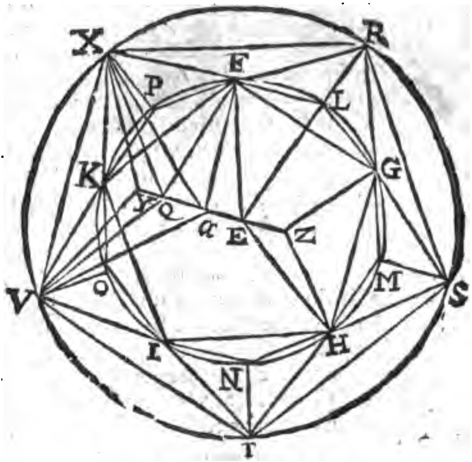
Coroll.

1. Hence it is manifeſt, that all the diameters of the cube are equal one to another, and do equally biſect one another in the center of the ſphere. And by the ſame means the right lines

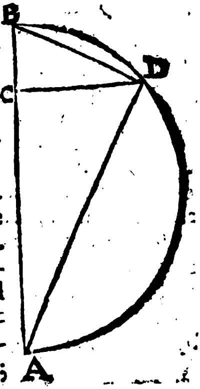
which conjoin the centers of the oppoſite ſquares are equally biſected in the ſame center.

2. The diameter of a ſphere contains in power the ſide of a tetraedron and of a cube, *viz.* ABq [k] = [l] BCq + [m] ACq. k 47.1. l 13.15. m 15.1

PROP. XVI.



To deſcribe an Icoſaedron ZGHIKFYVXRST, and encompaſs it in the ſphere, wherein were contained the foreſaid ſolids; and to demonſtrate that FG the ſide of the Icoſaedron is that irrational line which is called a minor line.



Upon AB the diameter of a ſphere deſcribe the ſemicircle ADB; and [a] make AB = 5 BC. then from C erect CD perpendicular, and draw AD and BD. At the diſtance EF = BD deſcribe the circle EFKNG; a 10. 6.

b 11.4. NG; *b* wherein inſcribe the equilateral pentagone
FKIHG. Divide equally in two parts the arches
FG,GH,*&c.* and join the right lines FL,LG,*&c.*
c 12.11. being the ſides of a decagone. Then *c* erect EQ,
LR,MS,NT,OV,PX equal to EF, and perpendi-
cular to the plane FKNG; and connect RS,ST,
TV, VX, XR; as alſo FX, FR, GR, GS, HS,
ST, HT, IT, IV, KV, KX. Laſtly, produce FQ,
and take QY = FL, and EZ = FL. and con-
ceive the right lines ZG, ZH, ZI, ZK, ZF to
be drawn; as alſo YV, YX, YR, YS, YT. Then
I ſay the Icoſaedron required is made.

d *conſtr.* For becauſe EQ,LR,MS,NT,OV,PX, are *d* e-
e 6.11. equal and *e* parallel, alſo thoſe lines that join them
f 33.1. EL,QR,EM,QS,EN,QT,EO,QV,EP,QX, *f* are
equal and parallel. And thence likewiſe LM (or
FG) RS,MN,ST,*&c.* are equal one to the other.
g 15.11. *g* therefore the plane drawn by EL, EM, *&c.* is
equidiſtant from the plane paſſing by QR, QS,
h 1.*def.*3. *&c.* *h* and the circle QXRSTV drawn from the
center Q is equal to the circle EPLMNO; and,
RSTVX is an equilateral pentagone. But EF,EG,
EH,*&c.* and QX,QR,QS,*&c.* being conceived to
k 47.1. be drawn; then becauſe FRq *k* = FEq + LRq, *l* or
l *conſtr.* EFq *m* = FGq, *n* therefore FR, FG, and ſo all
m 10.13. RS,FG,FR,RG,GS,GH,*&c.* ſhall be equal one
n *ſch.*48.1. to the other, and conſequently the ten triangles
and 1. *ax.* RFX,RFG,RGS, *&c.* are equilateral and equal.
o *cor* 14. Moreover, becauſe XQY is a *o* right angle, there-
11. fore XYq *p* = QXq + QYq *q* = VXq or FGq,
p 47.1. wherefore XY,VX. and likewiſe YV,YT,YS,YR,
q 10.13. ZG, ZH, *&c.* are equal. Therefore other ten
triangles are made equilateral and equal both
one to another, and to the ten former; and ſo
an Icoſaedron is made.

Moreover, divide equally EQ in *α*, draw the
r 15.*def.*1. right lines *α*F,*α*X,*α*V; and becauſe QX *r* = QV,
and *α*Q the common ſide, and FQX, EQV are
ſ 4.1. right angles, *ſ* therefore ſhall *α*X be = *α*V; and
by the ſame reaſon all the lines *α*X, *α*R, *α*S, *α*T,
*α*V,

αV, αF, αG, αH, αI, αK are equal. But becauſe t 9. 13.
ZQ.QE *t* :: QE.ZE. therefore Zαq *u* = 5 Eαq u 3. 13.
x = EQq (EFq) + Eαq *y* = αFq. therefore x 4. 2.
Zα = α F. *z* in like manner αF = Yα. there- y 47. 1.
fore the ſphere, whoſe center is α, and αF the z 15. 5.
ray, ſhall paſs by the 12 angular points of the
Icoſaedron.

Laſtly, Becauſe Zα. αE :: ZY. QE; *a* and ſo a 22. 6.
Zαq. αEq :: ZYq. QEq, *b* therefore ZYq = 5 b 14. 5.
QEq, or 5 BDq =: but ABq. BDq *c* :: AB.BC :: 5. c *cor.* 8. 6.
1. *d* therefore ZY = AB. *Which was to be done.* d 1. *ax.* 1.

Therefore if AB be put $\dot{\rho}$, *e* then EF = √ ABq e *ſch.* 12.
ſhall be alſo $\dot{\rho}$. and conſequently FG the ſide of 10.
the pentagone, and likewiſe of the Icoſaedron, *f* f 11. 13.
is a minor line. *Which was to be dem.*

Coroll.

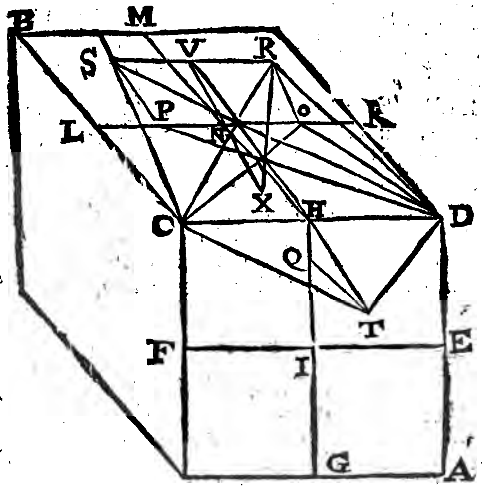
1. From hence is inferred, that the diameter of the ſphere is in power quintuple of the ſemidiameter of the circle encompaſſing the five ſides of the Icoſaedron.

2. Alſo it is manifeſt that the diameter of the ſphere is compoſed of the ſide of a hexagone, that is, of the ſemidiameter, and two ſides of the decagone of a circle encompaſſing the five ſides of the Icoſaedron.

3. It appears likewiſe that the oppoſite ſides of an Icoſaedron, ſuch as RX, HI, are parallels. For RX *a* is parall. to LP, *b* parall. to HI. a 33. 1.
b *ſch.* 26. 3.

PROP.

PROP. XVII.



To deſcribe a Dodecaedron, and comprehend it in the ſphere wherein the former figures were comprehended: and to demonſtrate that the ſide RS of the Dodecaedron is an irrational line of that ſort which is called an Apotome or reſidual line.

Let AB be a cube inſcribed in the given ſphere, and let all the ſides thereof be divided equally in the points E,H,F,G,K,L,*&c.* and join the right a 30. 6. lines KL,MH,HG,EF. *a* make HI.IQ :: IQ.QH; and take NO, NP, = IQ. then erect OR, PS, perpendicular to the plane DB, and QT to the plane AC, and let OR, PS, QT, be equal to IQ, NO, NP, whence DR,RS,SC,CT,DT, being connected, DRSCT ſhall be a pentagone of the dodecaedron required. For draw NV parallel to OR, and having drawn NV out as far as the center of the cube X, join the right lines DS,DO,DP,CR,

CP,HV,HT,RX. Becauſe DOq *a* = DKq (*b*KNq) a 47. 1.
+KOq *c* = 3 ONq (3 ORq) *d* thence DRq = b 7. *ax.* 1.
4 ORq *e* = OPq, or RSq. therefore DR = RS. c 4. 13.
By the ſame reaſon DR,RS,SC,CT, TP are equal. d 47. 1.
But becauſe OR *f* is = and *g* parallel to PS, e 4. 2.
therefore RS,OP, and conſequently RS, DC ſhall l *conſtr.* 96.
be alſo parallels. *h* therefore theſe with them 11.
that conjoin them DK,CS,VH, are in one and the g 33. 1.
ſame plane. Moreover, becauſe HI. IQ *h* :: IQ. h 9. 1.
(TQ.) QH *k* :: HN. NV. and both TQ,HN, and k 7. 11.
QH,NV *k* are perpendicular to the ſame plane, *l* l 6. 11.
and ſo likewiſe parallels, *m* THV ſhall be a right m 32. 6.
line. *n* therefore the Trapezium DRSC, and the n 1. *and* 2.
triangle DTS are in one plane extended by the 11.
right lines DC,TV. *o* therefore DCTSR is a pen- o 5. 13.
tagone, and that alſo equilateral, by what is
ſhewn already. Furthermore, becauſe PK. KN ::
KN.NP; and DSq *p* = DPq + PSq (PNq) = *p* p 47. 1.
DKq + PKq + NPq, *q* thence DSq = DKq + 3 KNq q 1. *ax.* 1.
= 4 DKq (4 DHq) *r* = DCq. therefore DS = DC. *and* 4. 13.
whence the triangles DRS, DCT, are equilateral r 4. 2.
one to another, *ſ* therefore the angle DRS = DTC, ſ 8. 1.
therefore the pentagone DTCSR is alſo equiangular. Moreover, becauſe AX, DX, CX, &c. are
ſemidiameters of the cube. *t* thence is XN = IH t 15. 13.
or KN, *u* and ſo XV = KP; wherefore becauſe u 1. *ax.* 1.
RVX, is a *x* right angle, *z* thence RXq = XVq + x 29. 1.
RVq (NPq) = KPq + NPq *a* = 3 KNq *b* = AXq z 47. 1.
or DXq, &c. therefore RX, AX, DX, and by the a 4. 13.
ſame reaſon XS,XT,AX, are equal one to another. b 15. 13.
And if by the ſame method, whereby the pentagone DTCSR was made, twelve like pentagones, touching the twelve ſides of the cube, be made, they ſhall compoſe a Dodecaedron; and a ſphere paſſing by their angular points, whoſe radius is AX or RX, ſhall comprehend that Dodecaedron. *Which was to be done.*

Laſtly, becauſe KN.NO *c* :: NO.OK, *d* thence c *conſtr.*
KL.OP :: OP.OK+PL. Therefore if AB the dia- d 15. 5.
meter of the ſphere be ſuppoſed *β*, then ſhall

KL

e 15. 13. f ſch. 12. 10. KL *e* = $\sqrt{\frac{AB}{3}}$ *f* be alſo ῥ. *g* whence OP or RS the ſide of the dodecaedron ſhall be a reſidual line. *Which was to be dem.*
g 6. 13.

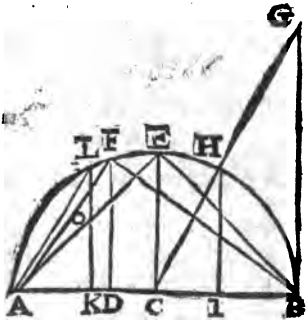
Coroll.

From this demonſtration it follows, 1. that if the ſide of a cube be cut in extreme and mean proportion, the greater ſegment ſhall be the ſide of the dodecaedron inſcribed in the ſame ſphere.

2. If the leſſer ſegment of a right line, cut in extreme and mean proportion, be the ſide of the dodecaedron, the greater ſegment ſhall be the ſide of the cube inſcribed in the ſame ſphere.

3. It is manifeſt alſo, that the ſide of the cube is equal to the right line which ſubtends the angle of a pentagone of the dodecaedron, inſcribed in the ſame ſphere.

PROP. XVIII.



To find out the ſides of the precedent five figures, and compare them together.

Let AB be the diameter of the ſphere given, and AEB the ſemicircle, and let AC *a* = $\frac{1}{2}$
a 10. 1. AB, and AD *b* = $\frac{1}{3}$
b 10. 6. AB then erect the perpendiculars CE, DF, and BG = AB. join AF, AE, BE, BF, CG; and let fall the perpendicular HI from H; and CK being taken equal to CI, from K erect the perpendicular
c 30. 6. KL, and join AL. Laſtly, *c* make AF. AO :: AO. OF.

Therefore 3. 2 *d* :: AB. BD *e* :: ABq. BFq the d *conſtr.*
ſide of a Tetraedron and 2. 1 :: *a* AB. AC :: ABq. e *cor.* 8. 6.
BEq. *f* the ſide of an Octaedron. f 14. 13.

Alſo 3. 1 *d* :: AB. AD *e* :: ABq. AFq. *g* the g 15. 13.
ſide of an Hexaedron. h *conſtr.*

Moreover, becauſe AF. AO *h* :: AO. OF. *k* k *cor.* 17.
thence ſhall AO be the ſide of a Dodecaedron. 13.
Laſtly, BG, (2 BC.) BC *l* :: HI. IC. *m* therefore l 4. 6.
HI = 2 CI *n* = KI. therefore HIq *o* = 4 CIq. m 14. 5.
q conſequently CHq *p* = 5 CIq. *q* therefore n *conſtr.*
ABq = 5 KIq. *r* therefore KI or HI is a ra- o 4. 2.
dius of a circle encloſing the pentagone of an p 47. 1.
Icoſaedron; and AK or IB *r* is the ſide of a de- q 15. 5.
cagone inſcribed in the ſame circle, *ſ* whence r *cor.* 16.
AL ſhall be the ſide of a pentagone, *t* and alſo 13.
the ſide of an Icoſaedron. Whereby it appears ſ 10. 13.
that BF, BE, AE are ρ̇ ⅂. and AL, AO ρ̇ ⅂, t 16. 13.
and BF ⊏ BE, and BE, AF, and AF ⊏ AO. And u 1. *ax.* 1.
becauſe 3 AFq = ABq *u* = 5 KLq, and AF × x 4. *ax.* 1.
AO ⊏ AF × OF, *x* and ſo AF × AO + AF × y 1. 2.
OF ⊏ 2 AF × OF, *y* that is, AFq ⊏ *z* 2 AOq. z 17. 6.
a thence ſhall 3 AFq (5 KLq) be ⊏ 6 AOq, a 47. 1.
conſequently KL ⊏ AO, and much rather AL
⊏ AO.

That we may expreſs the ſides in numbers; If AB be ſuppoſed √ 60, then, reducing what is already ſhewn to ſupputation, BF = √ 40, and BE = √ 30, and AF √ 20. Alſo AL = √: 30 − √ 180 (for AK = √ 15 − √ 3. and KL (HI) = √ 12:) Laſtly AO = √: 30 − √ 500 (√ 25 − √ 5.)

Schol.

Schol.

It is very apparent that beſides the five aforeſaid figures, there cannot be deſcribed any other regular ſolid figure (viz. ſuch as may be contained under ordinate and equal plane figures.)

a 21. 11.
b See ſchol. 32. 1.

For three plane angles at leaſt are required to the conſtituting of a ſolid angle; *a* all which muſt be leſs than four right angles, *b* but 6 angles of an equilateral triangle, 4 of a ſquare, and 6 of a hexagon, do ſeverally equal 4 right angles; and 4 of a pentagon, 3 of a heptagon, 3 of an octagon, *&c.* do exceed 4 right angles: Therefore only of 3, 4, or 5 equilateral triangles, of 3 ſquares or 3 pentagones, it is poſſible to make a ſolid angle. Wherefore beſides the five above-mentioned, there cannot be any other regular bodies.

Out of P. Herigon.

The Proportions of the ſphere and the five regular figures inſcribed in the ſame.

Let the diameter of the ſphere be 2. then ſhall

The Periphery or circumference of the greater circle, be 6, 28318.

The ſuperficies of the greater circle, 3, 14159.

The ſuperficies of the ſphere, 12, 56637.

The ſolidity of the ſphere, 4, 1879.

The ſide of the Tetraedron, 1, 62299.

The ſuperficies of the tetraedron, 4, 6188.

The ſolidity of the tetraedron, 0, 15132.

The ſide of the hexaedron, 1, 1547.

The ſuperficies of the Hexaedron, 8.

The ſolidity of the hexaedron, 1, 5396.

The ſide of the Octaedron, 1, 41421.

The ſuperficies of the octaedron, 6, 9282.

The ſolidity of the octaedron, 1, 33333.

The ſide of the Dodecaedron, 0, 71364.

The ſuperficies of the dodecaedron, 10, 51462.

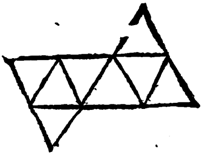
The ſolidity of the dodecaedron, 2, 78516.

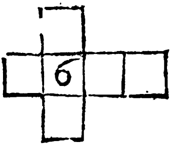
The ſide of the Icoſaedron, 1, 05146.

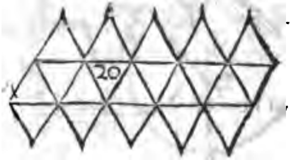
The ſuperficies of the Icoſaedron, 9, 57454.

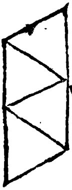
The ſolidity of the Icoſaedron, 2, 53615.

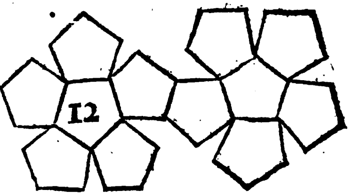
If five equilateral and equiangular figures, like theſe in the ſchemes beneath, be made of Paper, and rightly folded, they will repreſent the five regular bodies.







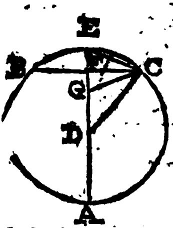




The End of the thirteenth Book.

THE FOURTEENTH BOOK OF EUCLIDE's ELEMENTS.

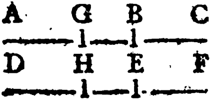
PROP. I.



A Perpendicular line DF drawn from D the center of a circle ABC to BC the ſide of a pentagone inſcribed in the ſaid circle, is the half of theſe two lines taken together, viz. of the ſide of the hexagone DE, and the ſide of the decagone EC inſcribed in the ſame circle ABC.

Take $FG = FE$, and draw CG: *a* Then CE a 4. 1.
is $= CG$. therefore the angle CGE *b* $= CEG$ b 5. 1.
b $= ECD$. therefore the angle ECG *c* $= EDC$ c 32. 1.
d $= \frac{1}{4} ADC$ *e* $= \frac{1}{2} CED$ $(\frac{1}{2} ECD)$ *f* conſequent- d *hyp.* *and*
ly the angle $GCD = ECG = EDC$. *g* wherefore 33. 6.
$DG = GC (CE.)$ therefore $DF = CE (DG) + EF =$ e 20. 3.
$\frac{DE + CE.}{2}$ *Which was to be dem.* f 7. *ax.*
g 6. 1.

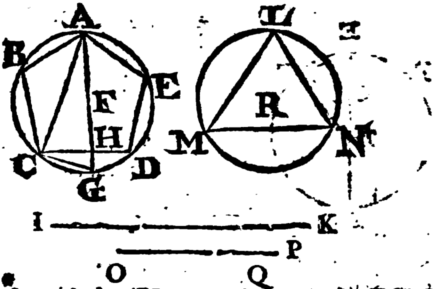
PROP. II.



If two right lines AB, DE, be cut according to extreme and mean proportion (AB. AG :: AG. GB. and DE. DH :: DH. HE.) they ſhall be cut after the ſame manner, viz. into the ſame proportions (AG. GB :: DH. HE.)

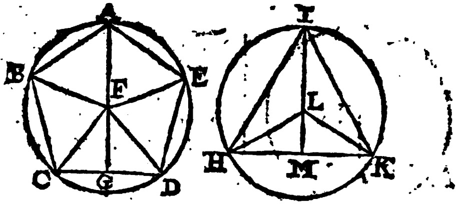
a 17. 6. Take BC=BG; and EF=EH. Then AB×BG
b 8. 2. is *a* = 5 AGq. wherefore ACq *b* = 4 ABG + AGq
c 2. *ax.* 1. *c* = BGq. In like manner ſhall DFq be = 5 DHq.
d 22. 5. & *d* therefore AC. AG :: DF. DH. whence by ad-
22. 6. dition AC + AG. AG :: DF + DH. DH. that
e 22. 5. is, 2 AB. AG :: 2 DE. DH. *e* conſequently AB.
f 17. 5. AG :: DE. DH; *f* whence by diviſion AG.
GB :: DH. HE. *Which was to be dem.*

PROP. III.



a *ſch.* 47. 1.
b 30. 6.
c 47. 1.
d 4. 2.
e 10. 13.
f 1, *and* *The ſame circle ABD comprehends both ABCDE*
3. *ax.* *the pentagone of a Dodecaedron, and LMN the trian-*
g 8. 13. *gle of an Icoſaedron inſcribed in the ſame ſphere.*
h 2. 13. & Draw the diameter AG. and the right lines
16. 5. AC, CG. and let IK be the diameter of the
k 22. 6. & ſphere, *a* and IKq = 5 OPq. *b* and make OP.
4. 4. OQ :: OQ. QP. Becauſe ACq + CGq *c* = AGq
l 15. 13. *d* = 4 FGq; and ABq *e* = FGq. *f* thence ACq
m *conſtr.* + ABq = 5 FGq. moreover, becauſe CA. AB *g* ::
n *cor.* 16. AB. CA — AB; and OP. OQ :: OQ. QP. *h* and
13. ſo CA. OP :: AB. OQ. *k* therefore 3 ACq (*l* IKq)
o 12. 13. 5 OPq (*m* IKq) :: 3 ABq. 5 OQq. therefore 3
p 10. 13. ABq = 5 OQq. But becauſe ML *n* is the ſide
q 15. 5. of a pentagone inſcribed in a circle, whoſe ra-
* *before* dius is OP, thence 15 RMq. *o* = 5 MLq *p* = 5
r 1. *ax.* 1. OPq + 5 OQq = * 3 ACq + 3 ABq *q* = 15
and ſch. FGq. *r* therefore RM = FG. *ſ* and conſequently
48. 1. the circle ABD is = to the circle LMN. *Which*
ſ 1. *def.* 3. *was to be dem.*

PROP. IV.



If from F the center of a circle encompaſſing the pentagone of a dodecaedron ABCDE, a perpendicular line FG be drawn to one ſide of the pentagone CD; the rectangle contained under the ſaid ſide CD and the perpendicular FG, being thirty times taken, is equal to the ſuperficies of a Dodecaedron. Alſo,

If from the center L of a circle incloſing the triangle of an Icoſaedron HIK, a perpendicular line LM be drawn to one ſide of the triangle HK, the rectangle contained under the ſaid HK, and the perpendicular LM, being thirty times taken, ſhall be equal to the ſuperficies of an Icoſaedron. a 8. 1.

Draw FA, FB, FC, FD, FE. *a* then ſhall the tri-
angles CFD, DFE, EFA, AFB, BFC be equal. but b 4. 1.
CD × FG *b* = 2 triangles CFD. therefore 30 c 15. 5.
CD × GF *c* = 60 CFD *d* = 12 pentagones d 6. ax.
ABCDE *e* = to the ſuperficies of a dodecaedron. e 17. 3.
Which was to be dem. f 41. 1.

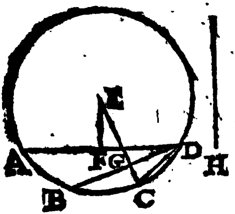
Draw LI, LH, LK; then HK × LM *f* is = g 15. 5.
2 triang. LHK. therefore 30 HK × LM *g* = 60 h 16. 13.
HLK = 20 HIK *h* = to the ſuperficies of an
Icoſaedron. *Which was to be dem.*

Coroll.

k 15. 5.

CD × FG. HK × LM *k* :: the ſuperficies of a dodecaedron to the ſuperficies of an Icoſaedron.

PROP. V.

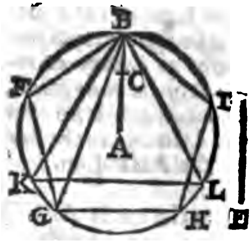


The ſuperficies of a Dodecaedron has to the ſuperficies of an Icoſaedron inſcribed in the ſame ſphere, the ſame proportion that H the ſide of a cube has to AD the ſide of an Icoſaedron.

Let the circle ABCD
a 3. 13. *a* incloſe both the pentagone of a dodecaedron, and the triangle of an Icoſaedron; whoſe ſides are BD, AD. upon which from the center E let fall the perpendiculars EF, EGC, and draw CD.

Becauſe EC + CD. EC *b* :: EC. CD. thence
b 9. 13. EG (*c* $\frac{1}{2}$ EC + CD.) EF (*d* $\frac{1}{2}$ EC) *e* :: EF. EG
c 1. 14. − EF. ($\frac{1}{2}$ CD.) but H. BD *f* :: BD. H − BD, *g*
d cor. 12. 13. therefore H. BD :: EG. EF. conſequently H × EF = BD × EG. wherefore ſince H. AD *h* ::
e 15. 5. H × EF. AD × EF. *k* thence ſhall be H. AD ::
f cor. 17. 13. BD × EG. AD × EF *l* :: the ſuperficies of a dodecaedron to the ſuperficies of an Icoſaedron.
g 2. 14. *Which was to be dem.*
h 1. 6.
k 7. 5.
l cor. 4. 14.

PROP. VI.



If a right line AB be cut in extreme and mean proportion, then as the right line BF, containing in power that which is made of the whole line AB, and that which is made of the greater ſegment AC, is to the right line E containing in power that which is made of the whole line AB, and that which is made of the leſſer ſegment BC; ſo is the ſide of the cube BG to the ſide of an Icoſaedron BK inſcribed in the ſame ſphere with the cube.

In the circle, whoſe ſemidiameter is AB, inſcribe BFGHI the pentagone of a dodecaedron,
and BKL the triangle of an Icoſaedron, *a* where- a *cor.* 17.
fore BG ſhall be the ſide of a cube inſcribed in 13.
the ſame ſphere. therefore BKq *b* = 3 ABq; and b 12. 13.
Eq *c* = 3 ACq. therefore BKq. Eq *d* :: ABq. c 4. 13.
ACq *e* :: BGq. BFq. wherefore by inverſion BGq. d 15. 5.
BKq :: BFq. Eq. *f* whence BG. BK :: BF. E. e 2. 14.
Which was to be dem. f 22. 6.

PROP. VII.

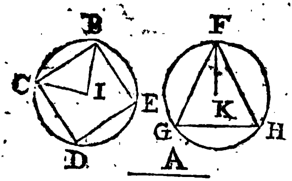
A Dodecaedron is to an Icoſaedron, as the ſide of a Cube is to the ſide of an Icoſaedron, inſcribed in one and the ſame ſphere.

Becauſe *a* the ſame circle comprehends both a 3. 14.
the pentagone of a dodecaedron, and the triangle
of an Icoſaedron, *b* the perpendiculars drawn b 47. 1.
from the center of the ſphere to the planes of the pentagone and triangle, ſhall be equal one to another. Therefore if the Dodecaedron and Icoſaedron be conceived divided into pyramides, right lines being drawn from the center of the

ſphere to all the angles, the altitudes of all the
pyramides ſhall be equal one to the other.
c 5. *and* 6. 12. Wherefore ſince the pyramides of equal heighth
with the baſes, and the ſuperficies of the dodecaedron is equal to twelve pentagones, and the
ſuperficicies of the Icoſaedron to twenty triangles, the dodecaedron ſhall be to the Icoſaedron,
as the ſuperficies of the dodecaedron is to the
d 5. 14. ſuperficies of the Icoſaedron, *d* that is, as the ſide
of the cube is to the ſide of the Icoſaedron.

PROP. VIII.

The ſame circle BCDE comprehends both the ſquare of the cube BCDE; and the triangle of the octaedron FGH inſcribed in one and the ſame ſphere.



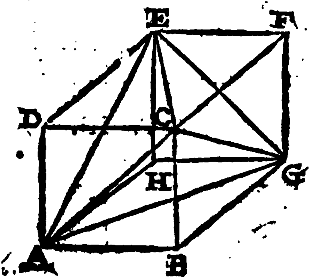
Let A be the diameter of the ſphere. Becauſe
a 15. 13. Aq $a = 3$ BCq $b = 6$ BIq; and alſo Aq $c = 2$
b 47. 1. GFq $d = 6$ KFq; thence ſhall BI be = KF. *e*
c 14. 13. therefore the circle CBED = GFH. *Which*
d 22. 13. *was to be demonſtrated.*
e 2. *def.* 1.

The End of the fourteenth Book.

THE

THE FIFTEENTH BOOK OF EUCLIDE's ELEMENTS.

PROP. I.



IN a cube given ABGHDCFE to deſcribe a pyramide AGEC.

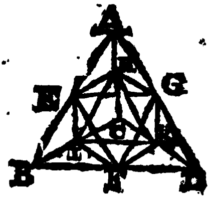
From the angle C draw the diameters CA, CG, CE; and connect them with the diameters AG, GE, EA. All which are *a* equal among themſelves, as being the diameters of equal ſquares: therefore the triangles CAG, CGE, CEA, EAG are equilateral and equal; and conſequently AGEC is a pyramide, which inſiſts upon the angles of the cube, and therefore *b* is inſcribed in it. *Which was to be done.*

a 47. 1.

b 31. *def.* 11.

PROP.

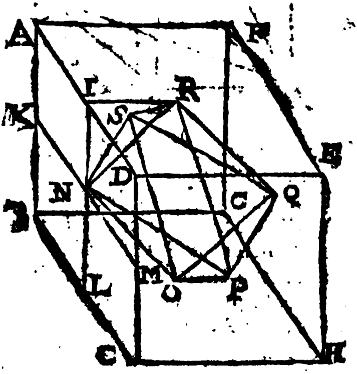
PROP. II.



In a pyramide given ABDC to deſcribe an octaedron EGKIFH.

a 10. 1. *a* Biſect the ſides of the pyramide in the points E, I, F, K, G, H, which join with the right lines EF, FG, GE, &c. All theſe are
b 4. 11. *b* equal one to the other; conſequently the 8 triangles EHI, IHK, &c. are equilateral and
c 27. def. 15. equal, and ſo make *c* an octaedron deſcribed *d*
d 31. def. 11. in the given pyramide. *Which was to be done.*

PROP. III.

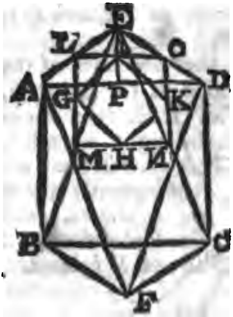


In a cube given CHGBDEFA to deſcribe an octaedron NPQSOR.

* 8. 4. Connect * the centers of the ſquares N,P,Q,S, O,R, with the twelve right lines NP,PQ,QS,&c.
a 4. 1. which are *a* equal among themſelves; and ſo
b 31. and make eight equilateral and equal triangles:
27. def. 11. wherefore *b* the Octaedron NPQSOR *b* is inſcribed in the cube. *Which was to be done.*

PROP.

PROP. IV.

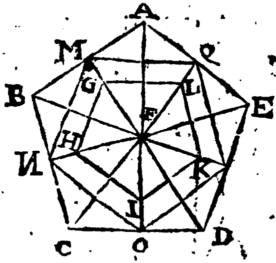


In Octaedron given ABCDEF, to inſcribe a cube.

Let the ſides of the pyramide EABCD, whoſe baſe is the ſquare ABCD, be equally biſected by the right lines LM, MN, NO, OL, which are *a* equal and a 4. 1.
b parallel to the ſides b 2. 6.
of the ſquare ABCD. *c* c 29. def. 1.
then the quadrilateral LMNO is a ſquare. In like manner, if the ſides of the ſquare LMNO be equally biſected in the points G, H, K, I, and GH, HK, KI, IG connected, GHKI ſhall be a ſquare. And if in the other 5 pyramides of the octaedron, the centers of the triangles be in the ſame ſort conjoined with right lines, then other ſquares will be deſcribed like and equal to the ſquare GHKI. wherefore ſix ſuch ſquares ſhall make a cube, which ſhall be deſcribed within an octaedron, *d* being its eight angles touch the d 31. def.
eight baſes of the octaedron in their centers. 11.
Which was to be done.

PROP.

PROP. V.



In an Icoſaedron given to inſcribe a Dodecaedron.

Let ABCDEF be a pyramide of the Icoſaedron, whoſe baſe is the pentagone ABCDE; and the centers of the triangles G,H,I,K,L; which connect with the right lines GH, HI, IK, KL, LG. Then GHIKL ſhall be a pentagone of the dodecaedron to be inſcribed.

For the right lines, FM, FN, FO, FP, FQ,
a *cor.* 3. 3. paſſing by the centers of the triangles, *a* do e-
b 4. 1. qually divide their baſes into two parts. *b* there-
c 4. 1. fore the right lines MN, NO, OP, PQ, QM *c*
d 8. 1. are equal one to the other; *d* whence alſo the
angles MFN, NFO, OFP, PFQ, QFM are e-
qual. therefore the pentagone GHIKL is equi-
e 4. 1. angular. *e* and conſequently equilateral, being
f 12. 13. FG, FH, FI, FK, FL *f* are equal. And if in
the other eleven pyramides of the Icoſaedron, the centers of the triangles be in like ſort conjoined with right lines, then will pentagones, equal and like to the pentagone GHIKL, be deſcribed. Wherefore 12 of ſuch pentagones ſhall conſtitute a dodecaedron; which alſo ſhall be deſcribed in the Icoſaedron, ſeeing the twenty angles of the dodecaedron conſiſt upon the centers of the twenty baſes of the Icoſaedron. Whereby it appears that we have deſcribed a dodecaedron in an Icoſaedron given. *Which was to be done.*

EUCLIDE's DATA.

A

Commentary or Preface written by the Philoſopher MARINUS, *on* EUCLIDE's *DATA.*

IN the firſt place we ought to ſet down (as a Foundation) what that is, which we call *DATUM* or GIVEN; then to conſider the Profit and Utility thereof; and in the third place, to what Art or Science this Tract doth appertain.

The Word *DATUM* therefore is diverſly defined, for the Antients have defined it after one manner, and later Writers after another, whence it follows that it ſeemeth a difficult thing to give a true Explication thereof; for ſome of them have not delivered the Definition of the Word; but have with much Labour and Trouble ſought certain Proprieties thereof, and ſome others collecting and mingling what hath been delivered by others before, have endeavored to define the Word *DATUM*; but not ſo exquiſitely but that they have contradicted themſelves; altho' what hath been ſaid by all of them, ſeems to be grounded on one and the ſame notion and ſuppoſition; for they all take the Word *DATUM* to be a thing compriſed; and therefore among ſuch as have endeavored to deſcribe it moſt ſimply, and with ſome ſimple difference, ſome of them have taken the Word *DATUM* to be the ſame with *ORDINATUM*, and ſo *Apollonius* underſtands it in his Tract of Inclinations, and in his univerſal Tract; and ſo others, as *Diodorus* takes it to be *COGNITUM* KNOWN; for in this Signification he takes the right line and the angles to be given, and all that may arrive to our Knowledge, altho' we may not be able well to expreſs it. But others have believed that it hath the ſame ſignification as the Word [*Effabile*] that may be declared, and ſo *Ptolomy* would have it, who calls thoſe things GIVEN, whoſe meaſure is known whether preciſely, or near the matter. Others alſo have thought the Word *DATUM* to be what is granted us by the Propoſer in the Hypotheſis; being that in the firſt Elements, a point given, and a right line given, is diverſly taken (that is to ſay, that who ſo would give and determine the quantity of a right line) all which things ſignify ſome COMPREHENSION; and therefore of all theſe Definitions, thoſe are moſt agreeable, which do moſt openly declare the COMPREHENSION, as we ſhall make evident by what follows.

Let us now unfold the diverſe Opinions of thoſe, who writing the nature of *DATUM* GIVEN, have not taken one ſimple Mark, or only Character for its Definition; and let us reduce it as in a Summary or Epitomy, to the end we may with the more eaſe know or number all their Differences. Some of them then have defined *DATUM* to be *Ordinatum* and *Porimon* together, and others *Ordinatum* and *Cognitum* together, and others *Porimon* and *Cognitum* together. Wherefore all ſeem to have ſo defined it as to have had regard to the *Comprehenſion*, or *Aſſuming* and *Invention* of the thing given; and to the end that we may the better conceive their Opinions, and that from the ſaying of many we may be able to draw a true Definition of what is propoſed, we will take notice in the firſt place of the Signification of all the ſimple Terms which they make uſe of, as alſo of the Terms oppoſed to them, to wit, *Inordinatum* and *Incognitum*, *Aporon* and *Irrational*; for thoſe things appertain to this Geometrical Buſineſs, to natural things, and to Mathematical Diſcipline.

Now we may call that *Ordinatum* (or Regulated) which doth always keep and obſerve that for which it is ſaid to be ordered, whether you regard its Magnitude or Species, or touching ſome other ſuch like thing: It is alſo thus defined, *Ordinatum* is that which cannot be done in divers manners, but in one only manner, and in ſome determined place: As for example, A right line drawn by two given points, is ſaid to be ordered, by reaſon it cannot be otherwiſe done, nor in divers manners. But an angle paſſing by two points is ſaid to be *Inordinatum* (or diſordinate and irregular) for that it is made in infinite and divers manners by a great or ſmall circle deſcribed by two points *ad infinitum*. Contrariwiſe, an angle conſtituted by three points, is ſaid to be *Ordinatum*, as alſo thoſe things which follow are ſaid to be *Ordinatum*, as to conſtitute an equilateral triangle on a right line, for it cannot be diverſly made, but unchangeably, on both the

given

extremities of the line. Again, To divide a given right line according to a given proportion, for that cannot be done but in one certain point. The things *Inordinatum* are ſuch as are done contrary to thoſe laſt mentioned; as to conſtitute a Scalene triangle, and to divide a right line indefinitely. Wherefore the Problem is ordered, is propoſed in the determination, conſidering that a certain thing may be in one manner ſaid to be *Ordinatum*, and *Inordinatum* in another, as an equilateral triangle, conſidering the equality of the ſides, it is *Ordinatum*, but conſidering its Magnitude it is *Inordinatum*, being in no wiſe determined.

But we call that *Cognitum* which is notorious, as clear and comprehended of us, and *Incognitum* that which is not known, or comprehended of us, as the length of a way is ſaid to be known, when we know how many Miles it contains; alſo that the three lines of a Rectiline triangle are equal to two right angles; and in like manner that the Binomial is Irrational, ſuch things are known, as alſo that it is only one right line that can touch a ſpiral line from a given point without it, from both parts; for if there were yet another line, two right lines would encloſe a ſpace, which is impoſſible. Again, irrational things are not ſaid to be unknown, but ſuch of them only which are neither known nor comprehended of us.

Porimon is that which we may make and conſtitute, that is to ſay, bring to our Underſtanding. Again, it is defined thus, *Porimon* is that which may be exhibited by Demonſtration, or which is apparent without Demonſtration, as to deſcribe a circle from a center and with a ſpace, as alſo to conſtitute not only an equilateral triangle, but alſo a Scalene; or to find a *Binomium*, or to find two right lines rational, commenſurable in power only, and other things which are known infinitely, are *Porimon*; as to deſcribe a circle by two points.

Aporon is wholly oppoſite to *Porimon*, as for example, the Quadrature of a circle, for that it hath not as yet been found; altho' it be certainly known that it may be: Nevertheleſs the manner of finding it out hath not been to this preſent comprehended. But we ſpeak here of that which is already known, which is called *Porimon principale*; for what hath not been as yet made, and yet nevertheleſs is poſſible, is called *Poriſton*, (or feaſible) altho' the Conſtruction be yet unknown. But *Aporon*, as hath been afore ſaid, is oppoſite to *Porimon*, and is that whoſe Nature is not as yet decided, nor well determined.

Effabile, that is to ſay, rational (or ſpeakable and explicable) is that whoſe Magnitude, Species, and Poſition, we may be able to declare; but this Definition is a little too general, for properly, and according to it ſelf, *Effabile* is that which is known by certain things, and according to a Meaſure given by Poſition, as of a ſpan, or a finger's breadth, *&c.*

Theſe things then being thus unfolded, we may eaſily perceive in what all thoſe things that we have afore ſpoken of do agree together, and wherein they do differ; and firſt of all how *Ordinatum* and *Cognitum* do agree together, and likewiſe their oppoſites the one to the other, for it cannot be ſaid that any one of thoſe things by counterchanging is the other, nor yet that the one hath not more extent than the other, altho' they agree in many things, as to deſcribe a right line by two points, and to conſtitute an equilateral triangle by three circles. But to ſquare a circle, that is indeed *Ordinatum*, yet nevertheleſs, *Incognitum*. Alſo that at a point of a ſpiral line there is but one touch line, that is, of the kind we call *Ordinatum*, and cannot be otherwiſe done, yet nevertheleſs the Demonſtration and Conſtruction thereof is not yet known. Again, the indefinite ſection, and the Conſtruction of the Scalenum is *Cognitum*; but is not *Ordinatum*; inſomuch as it is manifeſt, that amongſt thoſe things which are *Ordinatum*, there are ſome that are *Cognitum*, and others that are *Incognitum*; and contrariwiſe, that amongſt the things that are *Cognitum*, there are ſome that are *Ordinatum*, and others that are *Inordinatum*; and therefore thoſe things anſwer one another, as among Living Creatures, that which hath reaſon, with that which hath Feet, for there is no equality amongſt them, neither doth the one extend more than the other.

In like manner, *Ordinatum* and *Inordinatum* agree together, reſpecting *Porimon* and *Aporon*; ſeeing that between them there is a very great Reſemblance, and becauſe that they do differ only in the manner before expreſſed; for in truth the ſpiral line is *Ordinatum*; but it was not *Porimon* before *Archimedes*; and by the ſame reaſon thoſe things that are inordinate and known by an infinity of ways and means are *Porimon*, if any one ſhall undertake to invent their Conſtitution and Conſtruction. Yet nevertheleſs they are not ordinate, as to conſtitute a Scalenum triangle, it being no difficult thing to make known the Conſtitution thereof by an equilateral triangle, yea it is moſt eaſy, altho' it be inordinate and known by an infinity of ways.

And in the ſame manner do agree *Ordinatum* and *Inordinatum*, together with *Effabile* and *Irrationale*; for they agree together in many things, differing nevertheleſs by the foregoing reaſon, ſeeing thoſe things there mentioned bear no equality to each other, neither doth one thing contain the other; for all Binomiums and ſuch as are taken as Irrationals, are indeed ordinate, but yet they are not *Effabile*, or expreſſible, or to be unfolded, as the diameter of the ſquare is in reſpect of its ſide. Now touching *Effabile*, there are divers inordinate; becauſe they are diverſly known, and indeterminately, for a Scalenum triangle may be meaſured by a defined and propoſed meaſure, as explicable, altho' it be inordinate.

Now it is eaſy to ſee the agreement that there is between *Cognitum* and *Porimon*, but it is a difficult thing to expreſs or unfold their difference; foraſmuch as in their Natures they come ſo near to one another, as that there ſeems to be an equality between them: Nevertheleſs, there will ſome difference appear to him that ſhall conſider it more ſtrictly; for let it be conſented to that there can be only one line that can touch a ſpiral line in a certain point, that is *Cognitum*, yet notwithſtanding the Problem is not *Porimon*, it being not as yet comprehended; ſo as that all that which is *Cognitum*, is not therefore *Porimon*. But all that which is *Porimon* is alſo *Cognitum*, and therefore *Cognitum* appears to be of a greater extent than *Porimon*.

Now *Cognitum*, *Porimon*, and *Effabile*, do agree in ſome certain things, and do differ in other things by the ſame reaſon before alledged; for thoſe lines which are there called Irrationals are in truth known; and yet nevertheleſs are not *Effabile* or explicable. Contrariwiſe, every number is indeed *Effabile*, and yet every number is not *Cognitum*. But *Effabile* is always of its own nature expreſſible, altho' that ſome lengths may be now *Effabile*, and at another time not, if it be examined with ſome other according to one and the ſame meaſure. But alſo that ſame length is ſometimes known, and other times not, tho' they wholly agree with one another. Now it is a difficult thing to find ſomething that ſhall be *Effabile* and *Incognitum*, for *Cognitum* ſeems to be of a greater extent than *Effabile*, and by thoſe things it is manifeſt that *Porimon* and *Aporon* do differ from RATIONAL or *Effabile*, and from IRRATIONAL, for of IRRATIONAL ſome of them may be *Porimon*; but of RATIONAL none of them can be Irrationals; and therefore it is very eaſy to perceive in what the before expreſſed things agree. Notwithſtanding they ſeem to agree together, in ſuch ſort as that *Porimon* ſeems to be of a greater extent than *Effabile*.

Now by theſe things we may come to know the difference of theſe things that have been before ſpoken of, for in truth *Effabile* and *Irrationale* are ſo termed in reſpect of meaſure, which notwithſtanding is not as yet arrived to our Underſtanding, ſeeing that ſomething that is rational, may be as yet unknown to us, and in like manner may be rational, and yet may never be comprehended ſo to be. But *Ordinatum* and *Inordinatum* is ſo termed according to it ſelf, and according to the proper nature of the thing on which we contemplate, altho' it be not comprehended by us. As *Archimedes* had perceived ſome things to be ordinate from the nature of the things, the which *Serenus* had before contemplated. But *Cognitum* and *Incognitum* is ſpoken in reſpect of us, ſo as the things before mentioned do differ among themſelves; for theſe have reſpect to us, the others, ſome of them to their proper nature, and the reſt to meaſure.

Having then explained the agreements and differences of the things that have been propoſed, it remains now that we conſider what is meant by the Word *DATUM*, for of all thoſe that believe the Word *DATUM* to be that which is conſented to by the Propoſer in the Hypotheſis, are wide from what is ſought; becauſe that all the Elements of the things GIVEN are not compoſed of this ſort of GIVEN, which is according to the Hypotheſis, as may be ſeen in thoſe Tracts which have been made on this Subject GIVEN. Wherefore waving this Opinion, let us judge of the Definitions of others.

Then, that which is conſented to, or granted in the Hypotheſis, is ſomething which is conſequently known by the Principles; but ſuch as make uſe of Definitions of one only Word, do define it and remark it by ſome one of the before mentioned, as hath been ſpoken in the beginning, ſo as that almoſt all ſeem to have had this common notion of GIVEN, to wit, that it is comprehended even as the Word *DATUM*, doth alſo manifeſt it to be; and amongſt thoſe, theſe are the chief that do define it by the Hypotheſis or Suppoſition; and others have had regard to what is conſented to or granted. But we making uſe of the ſaid things as of a Rule and Direction to judge rightly, we may be able to find out a perfect Definition of *DATUM*; for it is certain that it ought to equal and be convertible with the thing defined, which is one thing proper to good Definitions. Now ſuch ſeems to be the Definition of the thing propoſed, which among the moſt ſimple and plain Expoſitors is defined *Porimon*, and amongſt the more acute, that which defineth it to be *Porimon* and *Cognitum* together; but all the reſt are imperfect; for that which defineth it *Ordinatum* is not ſufficient for the Comprehenſion and Knowledge of *DATUM*; becauſe that neither wholly ordinate nor alone ordinate, is not compriſed, ſeeing that there are things inordinate that have the ſame Condition, as hath been ſhewn. Again, that reaſon gives not Satisfaction, neither, which deſcribes it to be *Cognitum*; for all that is known is not comprehended, altho' that alone *Cognitum* be comprehended. Moreover, that alſo is not perfect which defineth it to be *Effabile*, for *Effabile* is not alone comprehended; ſeeing that ſome of the Irrationals are alſo comprehended. In like manner, all *Effabile* is not comprehended, as hath been before declared. Now amongſt the Definitions which expound it by the only Word, there remains that which defineth it to be *Porimon*, which ſeemeth greatly to manifeſt the Comprehenſion; for whole *Porimon* and alone *Porimon* is comprehended. Wherefore *EUCLIDE* himſelf uſeth ſuch a Definition in a Deſcription of all the kinds of GIVEN by him conceived and regarded. But amongſt ſuch Definitions as are compounded, that is a perfect Definition which defineth *DATUM* to be *Cognitum* and *Porimon* together, having *Cognitum* for analogical kind, and *Porimon* for difference; but that is imperfect which hath *Ordinatum* and *Porimon* together; for thoſe things which are ſuch, are not alone GIVEN, and that which defineth [illegible] *Ordinatum* and *Effabile* together, comprehendeth likewiſe the GIVEN, with the defect or want. But that of *Cognitum* and *Ordinatum* together, is not to be received or admitted, becauſe it doth exceed what is defined, for ſuch is not given alone: Therefore thoſe only which have declared that *DATUM* is *Cognitum* and *Porimon* together, ſeem to have attained the notion of GIVEN, for that which is ſuch is all, and alone comprehended, which two things ought to be in thoſe Definitions that are well given. But the former comes near to thoſe which have thus defined it: *DATUM* is that to which we may find an equal, according to thoſe things we have propoſed in the firſt Principles and Hypotheſis, of which number *EUCLIDE* is one, making uſe throughout of the Word πορίσασθαι, which ſignifies to exhibit or invent, altho' he leaves *Cognitum* as a Conſequent of *Porimon*; ſome one nevertheleſs might reprove him, for that in the firſt place he hath not defined *DATUM* in general; but immediately ſome of the kinds of GIVEN, altho' in his Elements of GEOMETRY, he hath defined the line ſimple before the Species or Kinds.

What is the Utility and Profit that ariſeth from this Tract of DATA, *or things GIVEN.*

AFter having explained univerſally, and according to what ſeemed neceſſary for our preſent Uſe, what this Word *DATUM* ſignifieth: It follows to ſhew the Utility of this Tract. Now this Tract is ſuch, as that it is not only ordained and inſtituted for its own reſpect, but for ſome other thing; for it is very neceſſary to a place which is called Reſolved, and we have already declared elſewhere how much ſtrength a reſolved place doth obtain in Mathematical Diſciplines, as alſo in Opticks and Cannons, which come very near to them, as well for that Reſolve is an Invention of the Demonſtration, as for that in ſuch like things it ſerves us much for the Invention of the Demonſtration, or for that it is much more excellent to meet with a Reſolutive Power, than to enjoy divers particular Demonſtrations.

To what Art or Science this Tract is referred.

NOW ſeeing the Conſideration of GIVEN is uſeful and profitable in all theſe kinds of Arts, for that it ſerveth much to RESOLUTION, it may well be ſaid to be recalled, not only to one only Science, but to the Mathematicks univerſally which treat of Numbers, Time, Swiftneſs, and ſuch like things, which treateth likewiſe of Reaſons, as alſo of Proportions, and in a word of all *Medietez:* Wherefore for the perfect and demonſtrative Knowledge of things GIVEN, being of ſo great Utility, *EUCLIDE* hath taken pains to frame this Book of things GIVEN, which Author amongſt all ſuch as have compoſed the Elements of Geometry, hath juſtly deſerved the firſt place and rank, and who having invented the Elements, or rather the Introductions almoſt of all Mathematical Diſciplines, to wit, of all Geometry in 13 Books, of Aſtronomy, of Muſick, and Opticks, he hath left in Writing the Elements RESOLUTIVE, in this Treatiſe of things GIVEN; but as he was a Geometrician, he hath particularly accommodated to Magnitudes, that was of the GIVEN; yet nevertheleſs common in other things, which Method hath alſo been obſerved by him, when treating univerſally of Reaſons and Proportions, he appropriates them to the Magnitudes mentioned in his Fifth Book of Planes.

Now it hath been declared in general what is the meaning of *DATUM*, to what Science it appertaineth, and how profitable the Contemplation thereof is. We will add to what hath been ſaid, the Deſcription of this Science which treats of things GIVEN; ſeeing that it is (as appears by what hath been ſaid) a Comprehenſion in all manners of things GIVEN; and of their Accidents and Proprieties. But having reſpect to the propoſed Book, we ſhall declare it to be an Elementary Doctrine of the whole Knowledge of things GIVEN, whence it follows that it will be very profitable, as alſo the things therein contained, ſeeing they refer to things GIVEN.

Now this Book is divided according to the Species or Kinds of the things GIVEN, and in the firſt Section are contained thoſe things which are given by Reaſon. Secondly, ſuch as are given by Poſition: and laſtly, ſuch as are given by Species or Kind; for that which is given by Magnitude is ſimple, and particularly contained in the others, and principally in thoſe things given by Species or Kind. Now he hath begun with thoſe things given by Reaſon and Poſition, foraſmuch as thoſe that are given by Species are conſtituted of them. *EUCLIDE* gives yet another Diviſion to this Book, for that he divides it into univerſal Magnitudes, Lines, and Superficies, and into circular Theoremes, which Order he hath alſo obſerved in the Definitions and Suppoſitions of this Book. Moreover, he uſeth a certain way of inſtructing, which proceeds not by Compoſition, but by Reſolution, as *Pappus* hath amply ſet down in his Commentaries on this Book.

The End of the PREFACE *of* MARINUS.

DEFINITIONS.

I. *PLanes or Spaces, Lines, and Angles, to which we may find others equal, are ſaid to be given by Magnitude.*

II. *A Reaſon is ſaid to be given, when we may find one of the ſame or equal thereto.*

III. *Rectiline figures, whoſe angles are given, and alſo the reaſon of the ſides to one another, are ſaid to be given by Species or Kind.*

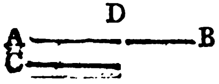
IV. *Points, Lines, and Angles, which have and keep always one and the ſame place and ſituation, are ſaid to be given by Poſition or Situation.*

V. *A Circle is ſaid to be given by Magnitude, when the ſemidiameter thereof is given by Magnitude.*

VI. *A Circle is ſaid to be given by Poſition, and by Magnitude when the center thereof is given by Poſition, and the ſemidiameter by Magnitude.*

VII. *Segments of Circles, whoſe angles and baſes are given by Magnitude, are ſaid to be given by Magnitude.*

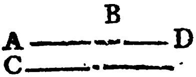
VIII. *Segments of a Circle, whoſe angles are given by Magnitude, and the baſes of the ſegments by Poſition and Magnitude, are ſaid to be given by Poſition and by Magnitude.*



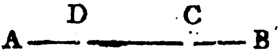
IX. *A Magnitude AB, is greater than another Magnitude C, by a given Magnitude BD, when having taken away the given*

given Magnitude DB, the reſt AD, is equal to the other Magnitude C.

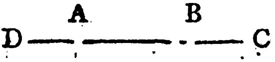
X. *A Magnitude AB, is leſs than another Magnitude C, by a given Magnitude BD, when having added thereto the given Magnitude BD, the whole AD is equal to the other magnitude C.*



XI. *A Magnitude AB, is ſaid to be greater than another magnitude CB, by a given magnitude AD, and in reaſon, when taking from the ſame magnitude the given magnitude AD, the reſt DB, hath to the other magnitude CB, a given reaſon.*



XII. *A magnitude AB is ſaid to be leſs than another magnitude BC, by a given magnitude AD, and in reaſon, when the given magnitude AD being added thereto, the whole DB hath to the other magnitude BC, a given reaſon.*



XIII. *A right line is ſaid to be drawn down from a given point, unto a right line given in Poſition, the right line being drawn in a given angle.*

XIV. *A right line is ſaid to be drawn up from a given point, to a right line given in Poſition, the right line beivg drawn in a given angle.*

XV. *A right line is againſt another right line in Poſition, when it is drawn parallel thereto through a given point.*

PROPO-

PROPOSITION I.

TWO magnitudes A and B, being given, the reaſon they have to one another A to B, is alſo given.

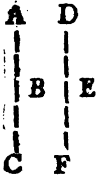
Demonſtration. For ſeeing that
a 1. *def.* the magnitude A is given, *a* we
may find one equal thereto, which let be C.
Again, foraſmuch as the magnitude B is given,
we may alſo find one equal to that, and let
that be D. Therefore ſeeing that A is equal
b 7. 5. to C, and B to D, as A is to C, *b* ſo is B to D,
c 16. 5. and by permutation, *c* as A ſhall be to B, ſo C
d 2. *def.* ſhall be to D. Therefore *d* the reaſon of A to
B is given, for it is the ſame reaſon as of C
to D, as we have found, and which ought to
be demonſtrated.

PROP. II.

If a given magnitude A, hath to ſome other magnitude B, a given reaſon, that other magnitude B, is alſo given by magnitude.

Demonſtr. For ſeeing that A is
given, we may find one equal
thereto, which let be C: And foraſmuch
a 2. *def.* as the reaſon of A to B,
is alſo given, we may find *a*
one of the ſame. Let it be
found, and let the reaſon be of C to D. Now
ſeeing that as A is to B, ſo C is to D;
and by permutation, as A is to C, ſo B is to
b 14. 5. D: But A is equal to C, therefore *b* B ſhall be
c 1. *def.* alſo equal to D. Therefore *c* the magnitude B
is given, ſeeing that thereto there hath been
found one equal, to wit, D.

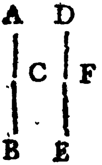
PROP. III.



If given magnitudes AB and BC, are compounded, that magnitude AC, that is compounded of them ſhall be alſo given.

Demonſtr. For ſeeing that AB is given, we may find one equal to it, which let be DE. Again, ſeeing that BC is given we may alſo find one equal to that, which let be EF. Wherefore ſeeing that DE is equal to AB, and EF is equal to BC, the whole AC *a* is equal to the whole DF. Therefore AC is given, ſeeing that DF is propoſed equal thereto. a 2. *ax.* 1.

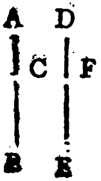
PROP. IV.



If from a given magnitude AB, there be taken away a given magnitude AC, the remaining magnitude CB is alſo given.

Demonſtr. Foraſmuch as AB is given, we may find one equal thereto, which let be DE. Again, ſeeing that AC is given, we may alſo find one equal to it, which let be DF. Seeing then that the magnitude AB is equal to the magnitude DE, and the magnitude AC to the magnitude DF; the reſt CB *a* ſhall be equal to the reſt FE. Wherefore CB is given, for to it there hath been found an equal, to wit, FE. a 3. *ax.* 1.

PROP. V.



If a magnitude AB, hath a given Reaſon to ſome part thereof AC, it will have alſo a given reaſon to the part remaining CB.

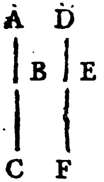
Demonſtr. Let DE be expoſed as a given magnitude, and ſeeing that the reaſon of the magnitude AB, to

a 2. *def.* to the magnitude AC, is given, *a* we may find
one of the ſame, which let be DE to DF;
therefore the reaſon of the ſame DE to DF is
b 2. *prop.* given, But DE being given, ſo is *b* alſo its
c 4. *prop.* part DF; and conſequently, *c* the reſt FE:
d 1. *prop.* Therefore *d* ſeeing that DE and FE are given,
the reaſon of the ſame DE to FE, is alſo
given. And foraſmuch as DE is to DF, as
AB is to AC, and by converſion, as DE to FE,
19. 5. ſo AB is to CB. But the reaſon of DE to FE is
given, as hath been demonſtrated; therefore
the reaſon of AB to CB is alſo given.

Scholium.

From this it is evident that if a magnitude hath to ſome part thereof a given reaſon, by diviſion, the reaſon that one part hath to the other, ſhall be alſo given. For ſeeing that as DE is to FE, ſo is AB to CB; by diviſion, as DF to FE, ſo AC to CB. But it hath been demonſtrated that the parts DF and FE are given, and conſequently their reaſon is alſo given: In like manner, therefore the reaſon of AC to CB is given.

PROP. VI.



If two magnitudes AB and BC, having to one another a given reaſon, are compounded, the magnitude AC compounded of them, ſhall allo have a given reaſon to each of them AB and BC.

Demonſtr. Let the given magnitude DE be expoſed, and ſeeing
that the reaſon of AB to BC is
given; let there be made one and the ſame of
the ſaid DE to EF; therefore the reaſon of the
a 2. *prop.* ſame DE to EF is given; and therefore *a* the
magnitude DE being given, both the one and
the other of them DE and FE, is given.
b 3. *prop.* Wherefore *b* the whole DF ſhall be alſo given.
There-

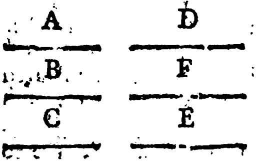
Therefore *c* the reaſon of the ſame DF to each c 1. *prop.*
of them DE and EF, ſhall be given. And foraſmuch then as AB is to BC, ſo is DE to EF; in compounding, *d* as AC is to BC, ſo is DF d 18. 5.
to EF: Therefore by converſion, as AC to AB, ſo is DF to DE. Therefore as the whole DF is to each of the other magnitudes DE and EF, ſo the whole AC is to each of the magnitudes AB and BC: Therefore *e* the reaſon of e 2. *def.*
the ſame AC to each of the magnitudes AB, and BC is given.

PROP. VII.

If a given magnitude AB, be divided according to a given reaſon AC to CB, each ſegment AC and CB is given.

Demonſtr. For ſeeing the reaſon of AC to CB is given, the reaſon of *a* AB to each of them (AC and CB) is alſo given. But AB is a 6. *prop.*
given: Therefore *b* each of the ſegments AC b 2. *prop.*
and CB is alſo given.

PROP. VIII.

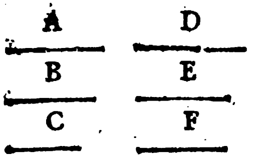


Magnitudes A and C, which have to one and the ſame a given reaſon B, ſhall be to one another in a given reaſon, A to C.

Demonſtr. For let the given magnitude D be expoſed, and ſeeing that the reaſon of A to B is given, let the ſame be done of the ſaid D to E. Now ſeeing that D is given, *a* E is alſo given. Again, ſeeing a 2. *prop.*
that the reaſon of B to C is given, let the ſame be done of E to F. But E is given, and therefore F is alſo given. But ſeeing that D is given,

b 1. *prop.* given, *b* the reaſon of the ſame D to F is given; and ſeeing that as A to B, ſo D to E, and as c 22. 5. B to C, ſo is E to F; in reaſon of equality, *c* as A is to C, ſo is D to F; but the reaſon of D to F is given. Therefore the reaſon of A to C is alſo given.

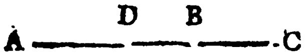
PROP. IX.



If two or more magnitudes A, B, and C, are to one another in a given reaſon, and that the ſame magnitudes A, B, and C, have to other magnitudes D, E, and F, given reaſons, altho' they be not the ſame, thoſe other magnitudes D, E, and F, ſhall be alſo to one another in given reaſons.

Demonſtr. Foraſmuch as the reaſon of A to B is given, as alſo that of A to D, the reaſon of D to B ſhall be given: But the reaſon of B to E is alſo given; therefore the reaſon of the ſame D to E ſhall be in like manner given. Again, ſeeing that the reaſon of B to C is given, and alſo that of B to E, the reaſon of E to C ſhall be given. But the reaſon of C to F is a 8. *prop.* alſo given. Therefore *a* the reaſon of E to F ſhall be given. But it hath been demonſtrated that the reaſon of D to E is alſo given; and b 8. *prop.* therefore *b* the reaſon of D to F ſhall be given: Therefore the magnitudes D, E, and F, are to one another in given reaſons.

PROP. X.

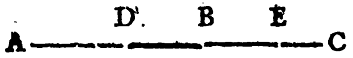


If a magnitude AB, be greater than another magnitude BC, by a given magnitude, and in reaſon, the magnitude

nitude AC compounded of both, ſhall be alſo greater than that ſame magnitude, by a given magnitude, and in reaſon: But if that compounded magnitude be greater than the ſame magnitude, by a given magnitude, and in reaſon; either the remainder ſhall be alſo greater than that ſame by a given magnitude, and in reaſon; or elſe the ſame remainder is given with the following, to which the other magnitude hath a given reaſon.

Demonſtr. For ſeeing that AB is greater than BC by a given magnitude, and in reaſon, let the given magnitude AD be taken away. Therefore *a* the reaſon of the remainder DB a 11. *def.* to BC is given; and in compounding, *b* the b 6. *prop.* reaſon of DC to BC is alſo given. But the magnitude AD is alſo given; therefore AC is greater than the ſame BC by a given magnitude, and in reaſon.

Again, Let the magnitude AC be greater than the magnitude BC, by a given magnitude, and in reaſon:

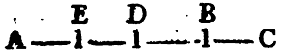


I ſay that the reſt AB, is either greater than the ſame BC by a given magnitude and in reaſon, or that the ſame AB, with that which followeth, to which BC hath a given reaſon, is given.

Foraſmuch as the magnitude AC is greater than the magnitude BC, by a given magnitude, and in reaſon, cut off from it the given magnitude: Now the ſame given magnitude is either leſs than the magnitude AB, or greater: Let it in the firſt place be leſs, and let it be AD. Therefore the reaſon of the remainder DC to CB is given. Wherefore by diviſion, the reaſon of DB to BC is given. But the magnitude AD is alſo given; therefore the magnitude AB is greater *c* than the magnitude BC by a c 11. *def.* given magnitude, and in reaſon. Now let the

given magnitude be greater than the magni-
tude AB, and let AE be put equal thereto;
d 11. *def.* therefore *d* the reaſon of the remainder EC to
e 5. *prop.* CB is given; and by converſion, *e* the reaſon
of the ſame BC to BE, is alſo given. But the
ſame EB with BA is given; for that the whole
AE is given: Therefore there is given AB,
with that which follows BE, to which BC
hath a given reaſon.

PROP. XI.



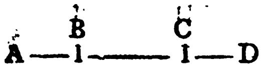
If a magnitude AB, be greater than a magnitude BC, by a given magnitude, and in reaſon, the ſame magnitude AB, ſhall be alſo greater than the magnitude compounded of them by a given magnitude, and in reaſon, and if the ſame magnitude be greater than the two others together by a given magnitude, and in reaſon, that ſame magnitude ſhall be alſo greater than the reſt by a given magnitude, and in reaſon.

Demonſtr. For ſeeing that the magnitude AB
is greater than BC by a given magnitude, and
in reaſon; let there be taken from it a given
a 11. *def.* magnitude AD: Therefore *a* the reaſon of the
b 6. *prop.* reſt DB to DC, is given, and therefore *b* the
reaſon of DC to BD ſhall be alſo given: Let
the ſame be done of AD to DE, therefore the
reaſon of the ſame AD to DE is given. But
c 2. *prop.* AD is given, therefore *c* DE is alſo given;
d 4. *prop.* and conſequently, *d* the reſt AE, is alſo given.
But ſeeing that as AD is to DE, ſo is DC to
e 16. 5. BD; by permutation, *e* as AD is to DC, ſo is
f 18. 5. DE to DB: Therefore by compounding, *f* as
AC is to CD, ſo is EB to DB; and by permu-
g 16. 5. tation, *g* as AC is to EB, ſo is DC to DB. But
the reaſon of DC to DB is given: Therefore alſo
is AC to EB, and conſequently, that of EB to
AC. But it hath been demonſtrated that AE is
h 11. *def.* given, therefore *h* AB is greater than AC by a
given magnitude, and in reaſon.

But now let AB be greater than AC by a given magnitude, and in reaſon: I ſay that the ſame AB is alſo greater than the reſt BC, by a given magnitude, and in reaſon.

For ſeeing that AB is greater than AC by a
given magnitude, and in reaſon. Let the given
magnitude AB be cut off there-from: There-
fore *i* the reaſon of the remainder EB to AC is i 11. *def.*
given, and conſequently, alſo ſhall be given
that of AC to EB. Let the ſame be done of
AD to DE, therefore the reaſon of AD to DE
is given; and by converſion, *k* the reaſon of k 5. *prop.*
AD to AE ſhall be alſo given, and conſequent-
ly that of AE to AD. Now AE is given,
therefore the whole AD *l* ſhall be alſo given; l 2. *prop.*
and ſeeing that as the whole AC is to the
whole EB, ſo the part cut off AD, is to the
part cut off ED, ſo alſo ſhall be *m* the remain- m 19. 5.
der DC to the remainder DB. But the reaſon
of AC to EB is given: Therefore alſo ſhall
be given that of DC to DB. Wherefore by
diviſion, *n* the reaſon of BC to DB is given; n *ſchol.* 5.
and conſequently alſo ſhall be given that of *prop.*
DB to BC. But it hath been demonſtrated
that AD is given: Therefore *o* AB is greater o 11. *def.*
than the ſame BC by a given magnitude, and
in reaſon.

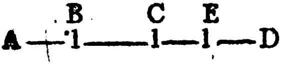
PROP. XII.



If there be three magnitudes AB, BC, and CD, and that the firſt AB, with the ſecond BC, to wit AC, be given. But the ſecond BC, with the third CD, to wit, BD, be alſo given: Either the firſt AB ſhall be equal to the third CD, or the one ſhall be greater than the other by a given magnitude.

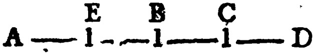
Demonſtr. Foraſmuch as each of the magnitudes AC and BD are given, the given magni-

tudes are either equal to one another, or unequal. Let them be firſt equal: Therefore AC is equal to BD, take away the common mag-
a 3. ax. 1. nitude BC, and there will remain *a* AB, equal to CD. But ſuppoſe them to be unequal, as in this ſecond figure, and let BD be greater than AC: Let then BE be put equal to AC. Now ſeeing that AC is given, BE is alſo given. But the whole BD is alſo
b 4. prop. given, the reſt ED *b* ſhall be ſo alſo; and foraſmuch as BE is equal to AC, taking away
c 3. ax. 1. the common magnitude *c* BC, there will remain AB equal to CE. But ED is given: Therefore CD is greater than AB by the given magnitude ED.

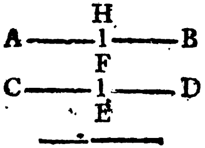


Scholium.

And if the firſt with the ſecond, to wit, AC, were greater than the ſecond with the third, to wit, BD, as in the other figure, CE would be made equal to the ſame BD, and by the ſame reaſons as was above demonſtrated, that AE is given and equal to CD; and therefore AB greater than CD by a given magnitude.



PROP. XIII.

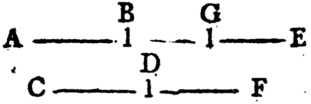


If there be three magnitudes AB, CD, and E, and that the firſt of them AB, hath a given reaſon to the ſecond CD; but the ſecond CD is greater than the third E, by a given magnitude, and in reaſon, alſo the firſt AB, ſhall be greater than

than the third E, by a given magnitude, and in reaſon.

Demonſtr. For ſeeing that CD is greater than E by a given magnitude, and in reaſon, let the given magnitude CF be taken there-from: Therefore the reaſon of the reſt FD to E is given. And foraſmuch as the reaſon of AB to CD is given, let the ſame be done of AH to CF. Therefore the reaſon of the ſame AH to CF is given. But CF is given: Therefore *a* AH is alſo given. And ſeeing that as the a 2. *prop.*
whole AB is to the whole CD, ſo the part cut off AH is to the part cut off CF, and ſo *b* alſo the reſt HB is to the reſt FD, the rea- b 19. 5.
ſon of the ſame HB to FD is alſo given. But the reaſon of FD to E is alſo given: Therefore *c* the reaſon of HB to E is given. But it c 8. *prop.*
hath been demonſtrated that AH is given: Therefore *d* AB is greater than the ſaid E by d 11. *def.*
a given magnitude, and in reaſon.

PROP. XIV.

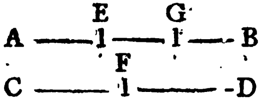


If two magnitudes AB and CD, have to one another a given reaſon, and that to each of them there be added a given magnitude, to wit, BE and DF; either the whole AE and CF ſhall have to one another a given reaſon, or the one ſhall be greater than the other by a given magnitude, and in reaſon.

Demonſtr. For ſeeing that each of thoſe magnitudes BE and DF, is given, *a* the reaſon of a 1. *prop.*
the ſaid BE and DF is alſo given; and if that reaſon be the ſame with that of AB to CD, that of the whole AE to the whole CF, *b* b 12. 5.
ſhall be the ſame; and therefore the reaſon of the ſaid AE to CF is given.

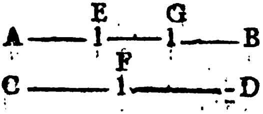
Now let the reaſon of BE to DF be not the ſame, with that of AB to CD, and let it be as AB to CD, ſo BG to DF. Therefore the reaſon of the ſaid BG to DF is given. But
c 2. *prop.* the magnitude DF is given, therefore *c* BG is alſo given; and ſeeing that the whole BE
d 4. *prop.* is given, *d* the reſt GE ſhall be alſo given. But foraſmuch as AB is to CD, as BG is to
e 12. 5. DF, *e* ſo alſo is the whole AG to the whole GF; and therefore the reaſon of the ſaid AG to CF is given: But the magnitude GE is
f 11. *def.* given: Therefore *f* the magnitude AE is greater than the magnitude CF by a given magnitude, and in reaſon.'

PROP. XV.



If two magnitudes AB and CD, have to one another a given reaſon, and that from each of them be taken away a given magnitude (to wit, from the magnitude AB the magnitude AE, and from the magnitude CD the magnitude CF) the remaining magnitudes EB and FD, either ſhall have to one another, a given reaſon, or the one of them ſhall be greater than the other by a given magnitude, and in reaſon.

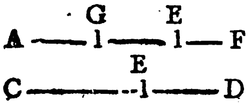
Demonſtr. For ſeeing that each magnitude AE and CF is given, the reaſon of AE to CF is given; and if it be the ſame with that of AB to CD, that of the remainder EB to the
a 19. 5. remainder FD, *a* ſhall be alſo the ſame; and therefore the reaſon of the ſaid EB to FD ſhall be alſo given.



But if it be not the ſame, let it be as AB to CD, ſo AC to CF. Now the reaſon of AB to CD is given;

therefore alſo that of AG to CF ſhall be given.
But CF is given, therefore *b* AG is given. b 2. *prop.*
But AE is alſo given, therefore *c* the reſt EG c 4. *prop.*
is given; and ſeeing that as AB is to CD, ſo
the part cut off AG is to the part cut off CF,
and ſo alſo is *d* the reſt GB to the reſt FD; the d 4. *prop.*
reaſon of the ſaid GB to FD is alſo given.
Therefore ſeeing that EG is given, EB is
greater than FD *e* by a given magnitude, and e 11. *def.*
in reaſon.

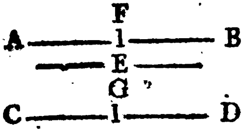
PROP. XVI.



If two magnitudes AB and CD, have to one another a given reaſon and that from one of them, to wit, CD, there be taken away a given magnitude DE; and to the other AB there be added a given magnitude BF, the whole AF ſhall be greater than the reſt CE, by a given magnitude, and in reaſon.

Demonſtr. For ſeeing that the reaſon of AB
to CD is given, let the ſame be made of BG
to DE; Therefore *a* the reaſon of the ſaid BG a 2. *def.*
to DE is given. But DE is given, therefore
b BG is alſo given. But BF is alſo given, b 2. *prop.*
therefore *c* the whole GF is given. And ſee- c 3. *prop.*
ing that as AB is to CD, ſo the part cut off
BG, is to the part cut off DE; and *d* ſo alſo d 19. 5.
is the remainder AG to the remainder CE;
the reaſon of the ſaid AG to CE is given:
But GF is given, therefore the magnitude AF
is greater than the magnitude CE by a given
magnitude, and in reaſon.

PROP. XVII.

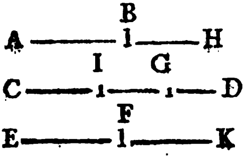


If there be three magnitudes AB, E, and CD, and that the firſt AB be greater than the ſecond E, by a given magnitude, and in reaſon. But the third CD be alſo greater than the ſame ſecond E, by a given magnitude, and in reaſon; the firſt AB ſhall have to the third CD, either a given reaſon, or elſe the one ſhall be greater than the other by a given magnitude, and in reaſon.

Demonſtr. For ſeeing that AB is greater than E by a given magnitude, and in reaſon, let the magnitude AF be taken away: Therefore the reaſon of the remainder FB to E is given. Again, ſeeing that CD is greater than the ſaid E by a given magnitude, and in reaſon, let the given magnitude CG be cut off therefrom; and the reaſon of the remainder GD to E ſhall be given: Therefore *a* the reaſon of FB to GD ſhall be alſo given. But to the ſaid FB and GD are added the given magnitudes AF and CG: Therefore the whole AB and CD *b* ſhall either have to one another a given reaſon, or the one ſhall be greater than the other by a given magnitude, and in reaſon.

8. *prop.*
b 14. *prop.*

PROP. XVIII.



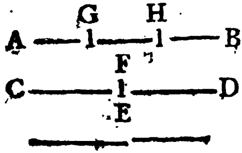
If there be three magnitudes AB, CD, and EF, and that the one of them, to wit, CD, be greater than either of the other AB or EF, by a given magnitude, and in reaſon; either the two others AB and EF, ſhall have

have to one another a given reaſon, or the one ſhall be greater than the other by a given magnitude, and in reaſon.

Demonſtr. Foraſmuch as the magnitude CD is greater than the magnitude AB by a given magnitude, and in reaſon, let the given magnitude DG be taken there-from: Therefore the reaſon of the remainder CG to AB is given. Let the ſame be made of GD to BH, therefore the reaſon of the ſaid DG to BH is given. But DG is given, therefore *a* BH is alſo given. a 2. *prop.* And ſeeing that as CG is to AB, ſo is GD to BH, *b* ſo alſo is the whole CD to the whole b 12. 5. AH, the reaſon of the ſaid CD to AH ſhall be alſo given.

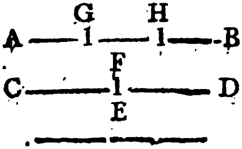
Again, ſeeing that the ſame CD is greater than EF by a given magnitude, and in reaſon; let the magnitude DI be cut off there-from: Therefore the reaſon of the remainder CI to EF is given: Let the ſame be made of DI to FK. Therefore the reaſon of the ſaid DI to FK ſhall be alſo given. But DI is given, therefore FK is alſo given. And ſeeing that as CI is to EF, ſo is ID to FK; ſo alſo is the whole *c* CD to the whole EK; the reaſon of c 12. 5. the ſaid CD to EK ſhall be given. But the reaſon of the ſame CD to AH is alſo given: Therefore *d* the reaſon of the ſaid AH to EK d 8. *prop.* ſhall be given. And ſeeing that from the ſaid AH and EK, the given magnitudes BH and FK are cut off, the magnitudes AB and EF *e* are either in a given reaſon to one ano- e 15. *prop.* ther, or the one is greater than the other by a given magnitude, and in reaſon.

PROP. XIX.



If there be three magnitudes AB, CD, and E, and that the firſt AB, be greater than the ſecond CD, by a given magnitude, and in reaſon; and that the ſecond CD be greater than the third E, by a given magnitude, and in reaſon; alſo the firſt magnitude AB ſhall be greater than the third E, by a given magnitude, and in reaſon.

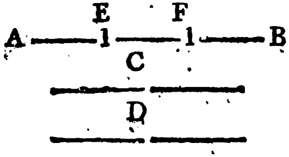
Demonſtr. For ſeeing that CD is greater than E by a given magnitude, and in reaſon; let the given magnitude CF be taken there-from: Therefore the reaſon of the remainder FD to E is given. Again, ſeeing that AB is greater than the ſame CD by a given magnitude, and in reaſon: Let the magnitude AG be taken there-from: Therefore the reaſon of the remainder GB to CD is given: Let the ſame be made of GH to CF: Therefore the reaſon of the ſaid GH to CF is given. But CF is given: Therefore alſo GH is given, and then AG is alſo given, the whole *a* AH ſhall be alſo given. But as GB is to CD, ſo is GH to CF, and ſo alſo *b* the remainder HB to the remainder FD: Therefore the reaſon of the ſaid HB to FD is given. But the reaſon of the ſame FD to E is alſo given: Therefore the reaſon of HB to E is in like manner given, and ſo is alſo the magnitude AE: Wherefore the magnitude AB *c* is greater than E by a given magnitude, and in reaſon.



a 3. *prop.*
b 19. 5.
c 11. *def.*

OTHER-

OTHERWISE.

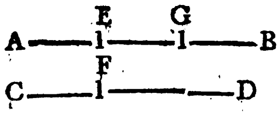


Conſtruction. Let there be three magnitudes AB, C, and D, and let AB be greater than C by a given magnitude, and in reaſon; but let C be alſo greater than D, by a given magnitude, and in reaſon: I ſay that AB is greater than D by a given magnitude, and in reaſon.

Demonſtr. Foraſmuch as AB is greater than C by a given magnitude, and in reaſon, let the given magnitude AE be cut off there-from: Therefore the reaſon of the remainder EB to C is given. But the magnitude C is greater than the magnitude D by a given magnitude, and in reaſon; therefore *d* EB is greater than D by a given magnitude, and in reaſon: Wherefore let the given magnitude EF be cut off there-from; and the reaſon of the remainder FB to D ſhall be given. But AF is *e* given, Therefore *f* AB is greater than D by a given magnitude, and in reaſon.

d 13. *prop.*
e 3. *prop.*
f 11. *def.*

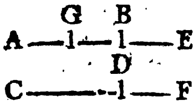
PROP. XX.



If there be two given magnitudes, AB and CD, and that from them there be taken magnitudes AE and CF, having to one another a given reaſon; either the remaining magnitudes EB and FD, ſhall have to one another given reaſons; or elſe the one ſhall be greater than the other by a given magnitude, and in reaſon.

Demonſtr. For ſeeing that both the magni-
tudes AB and CD, are given, the reaſon of the
a 1. *prop.* ſaid AB to CD is *a* alſo given; and if it be the
ſame as of AE to CF, that of the remainder
b 19. 5. EB to the remainder FD ſhall be *b* alſo the
ſame; and therefore the reaſon of the ſaid
EB to FD ſhall be alſo given. But if it be
not the ſame, let it be ſo as that AE be to
CF, as AG to CD. Now the reaſon of the
ſaid AE to CF is given: Therefore the reaſon
of the ſaid AG to CD is given. But CD is
c 3. *prop.* given, therefore *c* AG is alſo given. But the
d 4. *prop.* whole AB is likewiſe given, therefore *d* the
remainder BG is given. And ſeeing that as
AE is to CF, ſo is AG to CD, and alſo the re-
mainder EG to the remainder FD, the reaſon
of the ſaid EG to FD is given. But GB is
alſo given: Therefore the magnitude EB is
e 11. *def.* greater *e* than the magnitude BD by a given
magnitude, and in reaſon.

PROP. XXI.

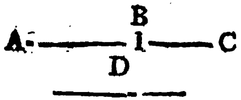


If there be two magnitudes given AB and CD, and to them be added other magnitudes BE and DF, having to one another a given reaſon: Either the whole AE and CF ſhall have to one another a given reaſon, or elſe the one ſhall be greater than the other by a given magnitude, and in reaſon.

Demonſtr. For ſeeing that both the magnitudes
a 1. *prop.* AB and CD are given, their reaſon *a* is alſo
given; and if it be the ſame reaſon as of BE
to DF, the reaſon of the whole AE to the
whole CF ſhall be alſo given; for it ſhall
b 12. 5. be *b* the ſame. But if it be not the ſame, let
it be as BE is to DF, ſo BG to CD: There-
fore the reaſon of the ſaid BG to CD is given.

But CD is given; therefore *c* alſo BG ſhall be c 2. *prop.*
given. But the whole AB is given, therefore
alſo the *d* remainder AG ſhall be given. And d 4. *prop.*
ſeeing that as BE is to DF, ſo is BG to CD,
and alſo *e* the whole GE to the whole CF, e 12. 5.
the reaſon of the ſaid GE to CF ſhall be likewiſe given. But AG is given, therefore the magnitude AE is greater than the magnitude CF by a given magnitude, and in reaſon.

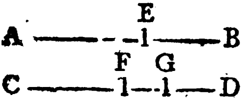
PROP. XXII.



If two magnitudes AB and BC, have to ſome other magnitude D, a given reaſon, alſo their compound magnitude AC, ſhall have to the ſame magnitude D, a given reaſon.

Demonſtr. For ſeeing that each magnitude AB
and BC, hath a given reaſon to D, the reaſon
a of AB to BC is given; and by compounding, a 8. *prop.*
b the reaſon of AC to BC is given. But that b 6. *prop.*
of BC to D is alſo given, therefore *c* the rea- c 8. *prop.*
ſon of the ſaid AC to D ſhall be likewiſe
given.

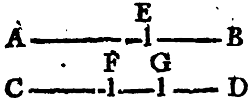
PROP. XXIII.



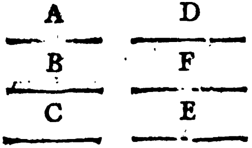
If the whole AB be to the whole CD in a given reaſon, and that the parts AE and EB, be to the parts CF and FD in given reaſons, altho' they be not the ſame, the whole (to wit, AB, AE, and BE,) ſhall be to the whole (to wit, CD, CF, and FD,) in given reaſons.

Demonſtr. For ſeeing that AE is to CF in a given reaſon, let the ſame be made of AB to CG; therefore the reaſon of the ſaid AB to

a 19. 5. to CG is given ; and conſequently, alſo that *a*
of the reſt EB to the reſt FG. But the reaſon
of FD to the ſame EB is alſo given : Therefore
b 8. *prop.* the reaſon of FD to FG *b* is likewiſe given ;
c 5. *prop.* and therefore *c* that of FD to the remainder
GD is alſo given. But the reaſon of AB to
each of the magnitudes CD and CG is given :
d 8. *prop.* Therefore *d* alſo the reaſon of CD to CG is
e 5. *prop.* given, and again *e* that of CD to the remainder
GD. But the reaſon of FD to DG is
f 8. *prop.* given, therefore alſo *f* that of the ſame CD to
FD, and conſequently
g 5. *prop.* that of *g* CD to the remainder
FC ; and therefore
alſo the reaſon of
CF to FD ſhall be given.
But the reaſon
of EB to FD is propoſed to be given ; therefore
the reaſon of CF to EB ſhall be given.
Again, for that the reaſon of AB to CD is
given ; and alſo that of CD to each of thoſe
FC and FD, the reaſon of the ſame AB to
h 8. *prop.* each of the ſaid FC and *h* FD, ſhall be likewiſe
given. But the reaſon of the ſaid FD
to EB is given : Therefore the reaſon of AB
to BE ſhall be alſo given, and conſequently
i 5. *prop.* AB to the remainder *i* AE. Wherefore by
k *ſch. 5. pr.* diviſion *k* the reaſon of AE to EB ſhall be likewiſe
given. But the reaſon of EB to FD is
given. Therefore alſo that of AE to FD. In
like manner, ſeeing that the reaſon of CD to
AB is given ; and that of AB to each of his
parts AE and EB ; alſo the reaſon of the ſaid
l 8. *prop.* CD to each of the ſaid AE and EB, *l* ſhall be
given : Wherefore each of the magnitudes AB,
CD, AE, EB, CF, and FD, is to each of the
others in a given reaſon.



PROP. XXIV.



If of three right lines A, B, and C, proportional, A to B, as B to C, the firſt A hath to the third C a given reaſon, it will alſo have to the ſecond B a given reaſon.

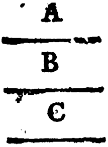
Demonſtr. For, let there be expoſed another
right line D, and ſeeing that the reaſon of A to
C is given: Let the ſame be made of D to F;
therefore the reaſon of D to F is given. But
D is given, therefore F is alſo given; betwixt
the two right lines D and F, let there be taken
a a mean proportional E. Therefore the rect- a 13.6.
angle made under D and F is equal *b* to the b 17.6.
ſquare of E. But the ſame rectangle of D
and F is *c* given: (for all the angles of that c 3. *def.*
rectangle are given, being right angles, and the
reaſons that the ſides have to one another are
alſo given;) therefore the ſquare of E is given,
and conſequently the ſame right line E is alſo
given (for one equal thereto may be found, *d* d 14.2.
ſeeing that the rectangle of D and F is given.)
But D is given, therefore *e* the reaſon of D to e 1. *prop.*
E is given, and as A is to C, ſo D is to F:
But as A is to C, *f* ſo the ſquare of A is to f 1.6.
the rectangle of A and C, and alſo as D is to
F, ſo the ſquare of D is to the rectangle of
D and F. Therefore as the ſquare of A is to
the rectangle of A and C, ſo the ſquare of D
is to the rectangle of D and F. But the rectangle of A and C is equal to the ſquare of B,
(ſeeing that A, B, and C, are proportional)
and that of D and F to the ſquare of E, therefore as the ſquare of A is to the ſquare of
B, ſo the ſquare of D is to the ſquare of E:

Where-

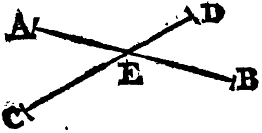
g 12. 6. Wherefore *g* as A is to B, ſo D is to E. But
h 2. def. the reaſon of D to E is given, therefore *h*
alſo the reaſon of A to B is given.

OTHERWISE.

Demonſtr. Foraſmuch as the reaſon of A to C is given, and that as A is to C, ſo the ſquare of A and C, the reaſon of the ſaid ſquare of A to the rectangle made of A and C, is alſo given. But to that rectangle made of A and C the ſquare of B is equal (ſeeing that A, B, and C, are proportional;) therefore the reaſon of the ſquare of A to the ſquare of B is given; and by conſequence, the reaſon of the line A to the line B is given; for to each of them A and B, we have exhibited an equal to the proper ſquare of each one.



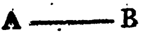
PROP. XXV.



If two lines AB and CD, given by poſition do interſect the point E in which they interſect one another, is given by poſition.

Demonſtr. For if it change its place, the one or the other of the lines AB and CD, would change its poſition: But ſo it is that by Suppoſition it changeth not: Therefore *a* the point
a 4. def. E is given by poſition.

PROP. XXVI.



If the extremities A and B, of a right line AB, be given by poſition, that ſame right line AB is given by poſition and by magnitude.

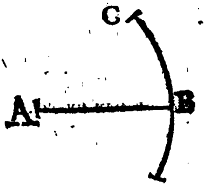
Demonſtr. For if the point A remaining in its place, the poſition, or the magnitude of the right line AB ſhall change, the point B will fall elſewhere. But ſo it is, that by Suppoſition it doth not fall elſewhere. Therefore the right line AB is given by poſition, and by magnitude.

PROP. XXVII.

If one of the extremities A of a right line AB, given by poſition and magnitude be given; the other extremity B ſhall be alſo given.

Demonſtr. For if the point A remaining in its place, the point B ſhall change and fall in ſome other place, either the poſition of the right line AB, or its magnitude would change: But ſo it is that according to the Suppoſition, neither the one nor the other doth change. Therefore the point B is given.

OTHERWISE.



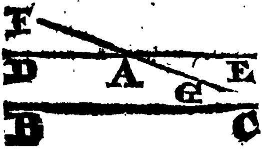
Conſtr. On the center A, with the diſtance AB, deſcribe the circumference BC.

Demonſtr. Therefore [a] that circumference BC is given by poſition. But the right line AB is alſo given by poſition; therefore the point [b] B is given.

a 6. *def.*

b 25. *prop.*

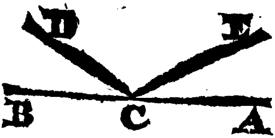
PROP. XXVIII.



If by a given point A, there be drawn a right line DAE, againſt another right line BC, given by poſition, the right line DAE ſo drawn, is given by poſition.

Demonſtr. For if it be not given, the point A remaining in its place, the poſition of the right line DAE may change: Let it then change if it be poſſible, and fall elſewhere, remaining parallel to BC, and let it be the line FAG: Therefore BC is parallel to the ſaid line FAG. But a 13. def. *a* the ſame BC is alſo parallel to DAE: b 30. 1. Therefore *b* DAE is parallel to the ſaid line FAG, which is abſurd; ſeeing they joyn together, and meet in A: Therefore the poſition of the right line DAE falls not elſewhere. Wherefore the ſaid line DAE is given by poſition.

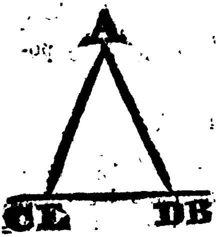
PROP. XXIX.



If to a right line AB, given by poſition, and to a point C given therein, there be drawn a right line CD, which ſhall make a given angle ACD, the line drawn CD, is given by poſition.

Demonſtr. For if it be not given by poſition, the point C remaining in its place, the poſition of the line CD obſerving the magnitude of the angle ACD, will fall elſewhere. Let it fall elſewhere then if it be poſſible, and let it be CE. Therefore the angle ACD is equal to the angle ACE, the greater to the leſſer; which is abſurd. Therefore the poſition of the right line CD, ſhall not fall elſewhere; and therefore the ſaid line CD is given by poſition.

PROP. XXX.



If from a given point A, be drawn to a right-line BC, given by poſition, a right line AD, making a given angle ADB, that line drawn AD is given by poſition.

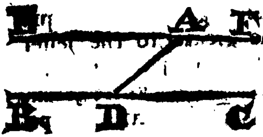
Demonſtr. For if it be not given, the point A remaining in its place, the poſition of the right line AD keeping, magnitude of the angle ADB, will change. Let it change then, and let it be the right line AE: Therefore the angle ADB is equal to the angle AEB, the greater *a* to the leſſer, which is abſurd. Therefore the poſition of the right line AD doth not change; and therefore the ſaid line AD is given by poſition.

a 16. 1.

OTHERWISE.

Conſtr. By the point A let there be drawn the line EAF, parallel to the right line BC.

Demonſtr. Then ſeeing that by the given point A, and againſt the right line BC, given by poſition, there is drawn the right line EF, thoſe lines EF and BC are parallels. But on the

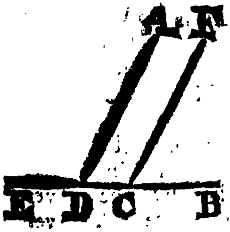


the ſame lines doth alſo fall the right line AD. Therefore b 29. 1. *b* the angle FAD is equal to the given angle ADB; and therefore it is alſo given. Wherefore to the right line EF given by poſition, and to the given point A therein, there is drawn the right line AD, making the given angle FAD. c 29. *prop.* Therefore *c* the ſaid line AD is given by poſition.

OTHERWISE.

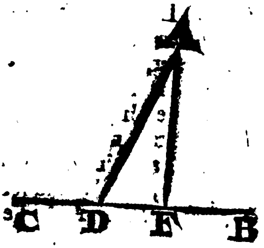
Conſtr. In the line BCE, let there be taken the given point C, and by the ſame let there be drawn the line CF, parallel to the ſaid DA.

Demonſtr. Foraſmuch as AD and FC are parallels, and that on them there doth fall the right line BCE, the angle FCB is equal *d* to d 29. 1. the given angle ADB; and therefore it is alſo given. And ſeeing that the right line BC is given by poſition, and that to a given point C therein, there is drawn the right line FC, making the given angle FCB, that ſame line FC e 29. *prop.* *e* is given by poſition. But by the given point A, oppoſite to the line FC given by poſition, there is drawn the line f 28. *prop.* AD. Therefore the ſaid line *f* AD is given by poſition.



OTHER-

OTHERWISE.



Conſtr. In the right line BC aſſume ſome point at F, and draw AF.

Demonſtr. Foraſmuch as each point A and F is given, the right line AF is given [g] by poſition. But the line BC is alſo given by poſition. Therefore * the angle AFD is given. But by ſuppoſition, the angle ADF is given: Therefore DAF (which is the reſidue [h] of two right angles) is given; and ſeeing that to the right line AF given by poſition, and to the given point therein A there is drawn the right line DA, making the given angle DAF, [i] that ſame line DA is given by poſition.

[g] 26. *prop.*

[h] 32. 1.

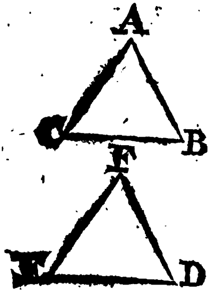
[i] 25. *prop.*

Scholium.

* *EUCLIDE ſuppoſeth here that two right lines being given by poſition, and inclining to one another do make a given angle, which ſome do demonſtrate after this manner.*

Demonſtr. Foraſmuch as the two right lines given by poſition, do incline to one another, the inclination of thoſe lines is given. But the angle is the inclination of the lines: Therefore the angle which makes the right lines given by poſition, and inclining to one another, is given.

Another thus demonſtrateth it.

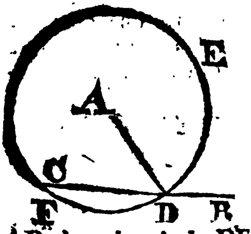


Conſtr. Let there be two right lines inclining to one another, as AB and CB, given by poſition, and in the line AB let there be taken a given point A, and in BC alſo ſome point, as C; and let the right line AC be drawn.

Demonſtr. Seeing that as well the point B, as each of the points A and C is given, *k* the three right lines AB, BC, and AC, are given by magnitude. Wherefore of three direct lines equal unto them, a triangle may be conſtituted: Let there then be made the triangle FDE, having the ſide FD equal to the ſide AB, the ſide FE equal to the ſide AC, and the baſe DE equal to the baſe BC. *k* 26. *prop.*

Seeing then the angles compriſed of equal right lines are equal, we have found the angle FDE equal to the angle ABC; and therefore the ſame *l* angle ABC is given. *l* 1. *def.*

PROP. XXXI.

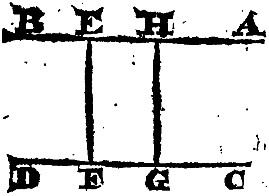


If from a given point A there be drawn to a right line given by poſition BC, a right line AD, given by magnitude, that line AD ſhall be alſo given by poſition.

Conſtr. From the center A, with the diſtance AD, let the circle DEF be deſcribed.

Demonſtr. Foraſmuch as the center A is given by poſition, and the ſemidiameter AD by magnitude, the circle DEF *a* is given by poſition. a 6. *def.* But the right line BC is alſo given by poſition: Therefore the point of interſection D *b* is given, b 25. *prop* and ſeeing that the point A is alſo given: *c* the right line AD is given by poſition. c 26. *prop.*

PROP. XXXII.

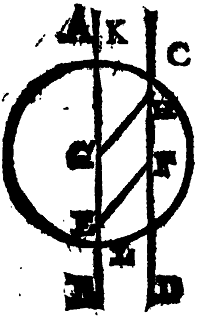


If unto parallel right lines AB and CD, given by poſition, there be drawn a right line EF, making the given angles BEF and EFD, the line drawn EF ſhall be given by magnitude.

Conſtr. For let there be taken in the line CD a given point G, and from that point let be drawn GH parallel to FE.

Demonſtr. Foraſmuch as the lines EF and HG are parallels, and that on them doth fall the line CD; *a* the angle EFD is equal to the a 29. 1. angle FGH. But the angle EFD is given, therefore the angle FGH is alſo given. And foraſmuch as to the right line CD given by poſition, and to the point G given in the ſame, there is drawn the right line GH, making the given angle FGH, *b* the ſaid line GH is given b 29. *prop* by poſition. But AB is alſo given by poſition, therefore *c* the point H is given. But the c 25. *prop.* point G is alſo given: Therefore *d* the line d 26. *prop.* GH is given by magnitude, and is *e* equal to e 34. 1. EF. Wherefore *f* the ſaid line EF is given f 1. *def.* by magnitude.

PROP. XXXIII.



If unto parallel right lines AB and CD, given by poſition, there be drawn a right line EF, given by magnitude, that line EF ſhall make the given angles BEF and DFE.

Conſtr. For let there be taken in the right line AB the point G, and by that point let there be drawn the line GH parallel to EF.

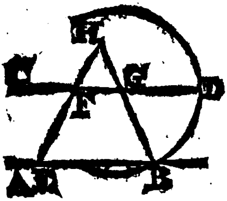
Demonſtr. Therefore EF
a 34. 1. is equal to the ſaid *a* GH. But EF is given by magnitude, therefore GH is alſo given by magnitude, But the point G is given, and therefore if on that point, with the diſtance
b 6. *def.* GH, there be deſcribed a circle, *b* that circle ſhall be given by poſition: Let it be then deſcribed, and let it be HKL, the ſaid circle HKL is therefore given by poſition. But the line CD which doth cut the circumference KHL in H, is alſo given by poſition. Therefore
c 25. *prop.* the ſaid point of interſection H *c* is given.
d 26. *prop.* But the point G is given: Therefore *d* the right line GH is given by poſition. But the right line CD is alſo given by poſition: There-
e *ſch.* 30. *prop.* fore *e* the angle GHF is given. But to that angle *f* the angle EFD is equal: Therefore
f 29. 1. the angle EFD is given; and therefore alſo the angle BEF; for that it is the reſidue of
g 29. 1. the ſumm of two *g* right angles.

OTHERWISE,

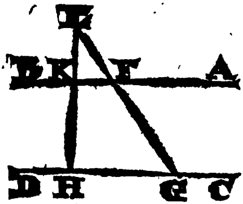
Conſtr. Let there be taken in the right line CD the point G, and let GD be put equal to

EF, then from the center G, with the diſtance GD, let there be deſcribed the circle DBH, and draw GB.

Demonſtr. Foraſmuch as the center G is given by poſition, and the ſemidiameter GD by magnitude, the circle BDH *h* is given by poſition. h 6. *def.* But the line AB is alſo given by poſition: Therefore *i* the point B is given. But the point i 25. *prop.* G is alſo given, therefore *k* the right line GB k 26. *prop.* is given by poſition. But the right line CD is alſo given by poſition: Therefore *l* the angle BGD is l *ſch.* 30. *prop.* given. Wherefore if EF be parallel to BG, the angle EFD *m* ſhall be m 29. 1. given, and conſequently alſo the other angle BEF. But the right lines BG and EF being not parallels, let them meet in the point H. Foraſmuch as EB is parallel to FG, and EF is equal to GD, that is to ſay, to BG; alſo FH *n* ſhall be equal to GH (for EH and BH n 14. 5. being cut proportionally *o* by the parallel FG, o 2. 6. as EF is to FH, ſo is BG to GH; and by permutation, as EF is to BG, ſo is FH to GH:) Therefore *p* the angle HFG is equal to p 5. 1. the angle HGF, but the ſaid angle HGF is given (for that it is equal *q* to the given angle q 15. 1. BGD:) Therefore the angle HFG is alſo given. But to that angle the angle BEF is equal; and therefore is given, as alſo the remaining angle EFG.



PROP. XXXIV.

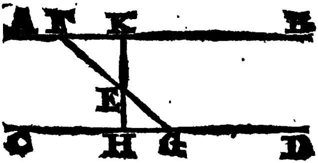


If from a given point E, there be drawn unto parallel right lines AB and CD, given by poſition, a right line EFG, that right line EFG ſhall be divided in a given reaſon (to wit) as EF to FG.

Conſtr. For from the point E let there be drawn the line EH, perpendicular to the line CD.

Demonſtr. Foraſmuch as from the given point
E there is drawn to the line CD the right
a 30. *prop.* line EH, making the given angle EHG *a* the
ſaid line EH is given by poſition, but both
the one and the other lines AB and CD is al-
b 25. *prop.* ſo given by poſition. Therefore *b* the points
of interſection K and H, are given. But the
c 26. *prop.* point E is alſo given: Therefore *c* each line
d 1. *prop.* EK and KH is given. Wherefore *d* the rea-
ſon of the ſaid EK to KH is given. But as
EK is to KH, ſo is EF to FG; (for in the
triangle GEH the line KF being parallel to
HG, the ſides EH and EG are cut proportio-
nally:) Therefore the reaſon of the ſaid EF
to FG is given.

OTHERWISE.



Conſtr. To the parallel right lines given by poſition, AB and CD, let there be drawn from the point E the right line FEG: I ſay that the reaſon of GE to EF is given.

Demonſtr. For from the point E let there be drawn to CD the perpendicular EH, and produced to the point K; ſeeing therefore that from the point E to the right line CD, given by poſition, there is drawn the line EH, making the given angle EHG, *a* the ſaid line EH is given by poſition. But each line AB and CD is alſo given by poſition: Therefore *b* each point of interſection H and K is given. But the point E is alſo given, therefore *c* each of the lines EH and EK is given by magnitude; and therefore *d* the reaſon of the ſaid EH to EK is given. But *e* as EH is to EK, ſo is EG to EF (for the oppoſite angles at the point E being equal, and the lines AB and CD parallels, the triangles EHG and EKF are equiangled; and therefore as EH is to EG, ſo is EK to EF; and by permutation as EH to EK, ſo is EG to EF.) Therefore the reaſon of the ſaid lines EG to EF is given.

a 30. *prop.*
b 25. *prop.*
c 26. *prop.*
d 1. *prop.*
e 4. 6.

PROP. XXXV.

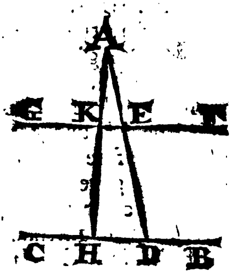
If from a given point A, to a right line BC, given by poſition, there be drawn a right line AD, which let be divided in E, in a given reaſon (to wit) as AE to ED, and that by the point of ſection E, there be drawn a right line FEG, oppoſite to the right BC, given by poſition, the line FG drawn ſhall be given by poſition.

Conſtr. For from the point A, let there be drawn the line AH, perpendicular to the line BC.

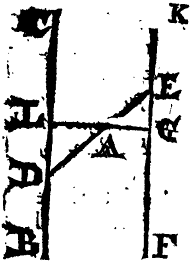
Demonſtr. For ſeeing that from the given point A there is drawn to BC given by poſition, the right line AH making the given angle AHD, *a* the ſaid line AH is given by poſition. But BC is alſo given by poſition; Therefore *b* the point H is given. But the point A is alſo

a 30. *prop.*
b 25. *prop.*

c 26. *prop.* given; Therefore *c* the line AH is given by magnitude and by poſition. And ſeeing that *d* 2. 6. *d* as AE is to ED, ſo is AK to KH, and that the reaſon of AE to ED is given, alſo the reaſon of AK to KH is given; *e* 6. *prop.* and by compounding, *e* the reaſon of AH to AK is given. But AH is given by magnitude: *f* 2. *prop.* Therefore *f* alſo AK is given by magnitude. But AK is alſo given by poſition, and the point A is given: There- *g* 27. *prop.* fore *g* the point K is alſo given, and ſeeing that by the ſaid given point K there is drawn the line FG, oppoſite to the right line BC *h* 28. *prop.* given by poſition; the ſaid line FG *h* is given by poſition.



PROP. XXXVI.



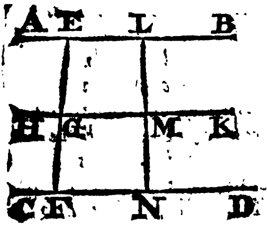
If from a given point A, there be drawn to a right line BC given by poſition, a right line AD, and to it be added a right line AE, having to the ſame AD a given reaſon, and that by the extremity E of the added line AE, there be drawn a right line FEK, oppoſite to the line BC, given by poſition, that ſame line FEK ſhall be given by poſition.

Conſtr. For from the point A let there be drawn to the line BC, the perpendicular AL, and let it be prolonged to the point G.

Demonſtr. Foraſmuch as from the given point A, there is drawn to the right line BC, given by poſition, the right GL, which makes the given angle GLD, *a* that line GL is given by poſition. But BC is alſo given *b* by poſition, therefore *b* the point L is given; and ſeeing that the point A is alſo given, the line *c* AL is given. But foraſmuch as the reaſon of AE to AD is given, and that *d* as the ſaid AE is to AD, ſo is AG to AL; (becauſe the triangles ALD and AGE are equiangled) the reaſon of AG to AL is alſo given. But AL is given by magnitude: Therefore *e* AG is given by magnitude. But it is alſo given by poſition, and the point A is given: Therefore *f* the point G is alſo given. And ſeeing that by the ſame given point G, there is drawn the line FK, oppoſite to the right line BC, given by poſition, *g* the ſaid line FK is given by poſition.

a 30. *prop.*
b 26. *prop.*
c 26. *prop.*
d 4. 6.
e 2. *prop.*
f 27. *prop.*
g 28. *prop.*

PROP. XXXVII.



If unto parallel right lines AB and CD, given by poſition, there be drawn a right line EF, divided in the point G, in a given reaſon, (to wit, of EG to GF;) but if by the point of ſection G, there be drawn oppoſite to the right lines AB or CD, given by poſition, a right line HGK, that line drawn ſhall be given by poſition.

Conſtr. For let there be taken in the line AB the given point L, and from that point let there be drawn the line LN, perpendicular to CD.

Demonſtr. Seeing that from the given point L, there is drawn to the right line CD, the line LN, making the given angle LND, the ſaid

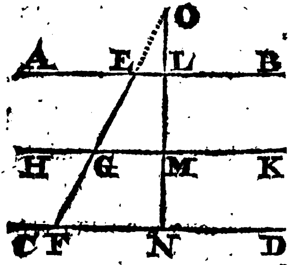
a 30. *prop.* ſaid LN *a* is given by poſition. But CD is
alſo given by poſition: Therefore the point
b 25. *prop.* N *b* is given. But the point L is alſo given:
c 26. *prop.* Therefore *c* the line LN is given; and ſeeing
that the reaſon of FG to GE is given, and
that * as FG is to GE, ſo is NM to ML;
the reaſon of the ſaid MN to ML is given;
d 6. *prop.* and in compounding, *d* the reaſon of LN to
LM is alſo given. But LN is given by mag-
e 2. *prop.* nitude, therefore ML is *e* given by magnitude.
But it is alſo given by poſition, and the point
f 27. *prop.* L is given: Therefore the point M *f* is alſo
given. And conſidering that by the ſaid point
M there is drawn the right line KH, oppoſite
to the right line CD, given by poſition, the
ſaid line KH is alſo given by poſition.

Scholium.

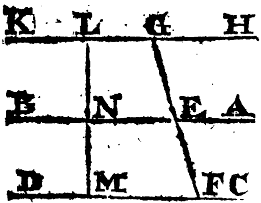
** EUCLIDE ſuppoſeth here, that as FG is to GE, ſo NM is to ML; but by another it is thus demonſtrated.*

The lines EF and LN are parallels or not parallels: Let them in the firſt place be parallels, and foraſmuch as by Conſtruction the lines EL, FN, EF, and LN, are parallels, EN ſhall be a parallelogram: and therefore the ſide EF is equal to the ſide LN Again, ſeeing that MG is parallel to FN, and GF to MN, GN ſhall be alſo a parallelogram; and therefore the ſide GF is equal to the ſide MN. Wherefore the equal ſides EF and LN, ſhall have to the equal ſides FG and
g 7. 5. *MN,* g *one and the ſame reaſon. Therefore as EF is to FG, ſo is LN to MN; and in dividing,*
h 17. 5. h *as GE to GF, ſo is LM to MN.*

*Now ſuppoſe that the lines EF and LN be not
parallels, but that they meet in the point O. Foraſmuch
as in the
triangle OFN there
is drawn HK, parallel
to FN one of
the ſides; i the ſides* i 2. 6.
*OF and ON are divided
proportionably;
and therefore as FG
is to GO, ſo is NM
to MO. Again, ſeeing
that in the triangle
OGM there is
drawn EL, parallel to the ſide GM, the ſides
OG and OM are divided proportionally: Wherefore
k as OE is to EG, ſo is OL to LM; and by compounding,* k 2. 6.
*l as OG is to EG, ſo is OM to LM; but
it hath been demonſtrated that as FG is to GO, ſo* l 18. 5.
is NM to MO; therefore in reaſon of equality, m m 22. 5.
as FG is to GE, ſo is NM to ML.



PROP. XXXVIII.



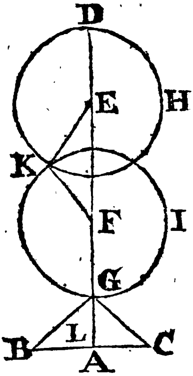
If unto parallel right lines AB and CD, there be drawn a right line EF, and that to it there be added ſome other right line EG, which hath a given reaſon to the ſame EF, but if by the extremity G of the added line EG, there be drawn a right line HK, againſt the parallels given by poſition AB and CD, the line drawn HK ſhall be alſo given by poſition.

Conſtr. For let there be taken in the line AB, the given point N, and from thence let there be drawn to CD the perpendicular NM, and let it be prolonged to the point L.

Demonſtr. Foraſmuch as from the given point N there is drawn to the right line CD, given by poſition, the right line NM making a given angle NMF, the ſaid angle NMF *a* is given by poſition. But the line CD is alſo given by poſition: Therefore *b* the point M is given. But the point N is alſo given: Therefore *c* the line NM is given, and for that the reaſon of EG to EF is given, and that *d* as EG is to EF, ſo is LN to NM, the reaſon of LN to NM is alſo given: But NM is given, therefore LN is *e* alſo given. But the point N is given: Therefore *f* the point L is alſo given. Seeing then that by the given point L there is drawn the right line HK, oppoſite to the line AB given by poſition, *g* the ſaid line AK is alſo given by poſition.

a 30. *prop.*
b 25. *prop.*
c 26. *prop.*
d *ſch.* 37. *prop.*
e 2. *prop.*
f 27. *prop.*
g 28. *prop.*

PROP. XXXIX.



If all the ſides of a triangle ABC are given by magnitude, the triangle is given by Kind.

Conſtr. For, let there be expoſed the right line DG given by poſition, ending in the point D; but being infinite towards the other part G, and therein let there be taken DE, equal to AB.

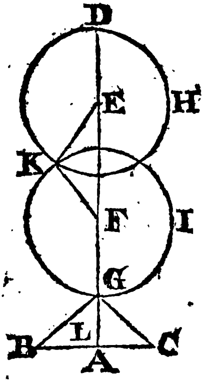
Demonſtr. Now ſeeing the ſaid AB is given by magnitude, DE is ſo alſo; but the ſame DE is alſo given by poſition, and the point D is given: Therefore *a* the point E is given.

a 27. *prop.*

Again, Let EF be put equal to BC; and ſeeing that BC is given by magnitude, EF ſhall be ſo alſo. But the ſaid EF is in like manner given by poſition, and the point E is given: Therefore *b* the point F is given. b 27. *prop.*

Furthermore, Let FG be taken equal to AC. Now foraſmuch as the ſaid AC is given by magnitude, FG is ſo alſo. But FG is alſo given by poſition, and the point F is given: Therefore the point G is alſo given. Now from the center E, with the diſtance ED, let there be deſcribed the circle DHK, *c* c 6. *def.* and that circle ſhall be given by poſition. Again, on the center F, and diſtance FG, let there be deſcribed the circle GLK. Therefore *d* the ſaid circle GLK is given by poſi- d 6. *def.* tion; and therefore *e* the point of interſection e 25. *prop.* K is given. But each of the points E and F is given: Therefore each line *f* EK, EF, and f 26. *prop.* FK, is given by poſition and magnitude. Therefore the triangle EKF is given * by kind; but it is equal and alike to the triangle ABC; and therefore the triangle ABC is alſo given by kind.

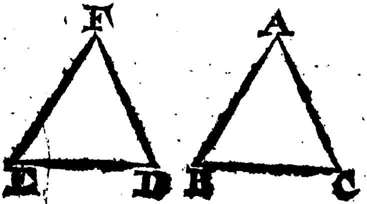
Scholium.



* *EUCLIDE ſuppoſeth here that a triangle whoſe ſides are given by magnitude and poſition, is given by kind; but the antient Interpreters demonſtrate it in a manner thus. Foraſmuch as the right lines KE and EF are given,* g g 1. *prop.* *the reaſon which they have to one another is given. Alſo the right lines EF and FK being given, their reaſon is alſo given; and in like manner, the reaſon*

of the ſaid EK and FK is given. Again, ſeeing that the ſame lines KE and EF are given by h ſch. 30. prop. *poſition,* h *the angle KEF is given by magnitude: Moreover, the right lines EF and FK being given by poſition, the angle EFK is given by magnitude, as is alſo the reſidue EKF, and ſo in the triangle EKF are all the angles given, and alſo the rea-* i 5. def. *ſons of the ſides: Therefore* i *the ſaid triangle EKF is given by kind.*

PROP. XL.



If the angles of a triangle ABC, are given by magnitude, the triangle is given by kind.

Conſtr. Let there be expoſed the right line DE, given by poſition and by magnitude; and let there be conſtituted at the point D the angle EDF, equal to the angle CBA; but in the point E the angle DEF, equal to the angle BCA; therefore the third angle BAC is equal to the third angle DFE.

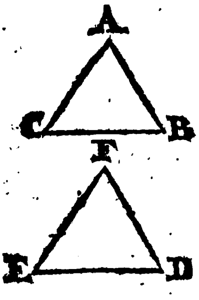
Demonſtr. For each of the angles conſtituted in the points A, B, and C, is given: Therefore each of thoſe which are poſited in the points D, F, and E, is alſo given; and ſeeing that to the right line DE given by poſition, and to the point D given therein, there is drawn the right line DF, which makes the gi- a 29. prop. ven angle EDF, *a* the line DF is given by poſition; and by the ſame reaſon, the line b 25. prop. EF is given by poſition: Therefore *b* the point F is given by poſition. But each of the c 26. prop. points D and E is given: Therefore *c* each of the lines DF, DE, and EF, is given by magnitude. Wherefore the triangle DFE is given by kind; and is alike to the triangle ABC: Therefore the triangle ABC is given by kind.

PROP. XLI.

If a triangle ABC, hath one angle BAC given, and that the two ſides BA and AC, which do conſtitute it, have to one another a given reaſon, the triangle is given by kind.

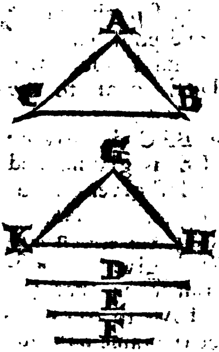
Conſtr. For, let there be expoſed the right line DF given by magnitude and poſition. But thereon, and at the given point F, let there be conſtituted the angle DFE equal to the angle BAC.

Demonſtr. Now the angle BAC is given: Therefore alſo the angle DFE is given, and ſeeing that to the right line DF given by poſition, and from the given point F therein, is drawn a right line FE, making the given angle DFE, *a* the ſaid line FE is given by poſition. a 29. *prop.* But ſeeing that the reaſon of AB to AC is given, let the ſame be made of DF to FE, then let DE be drawn. Therefore the reaſon of DF to FE is given. But DF is given: Therefore *b* FE is given by b 2. *prop.* magnitude. But the ſame FE is alſo given by poſition, and the point F is given. Therefore *c* the c 27. *prop.* point E is alſo given. But each of the points D and F is given: Therefore *d* each of the right lines d 26. *prop.* DF, FE, and DE, is given by poſition and magnitude. Wherefore *e* the triangle DEF is e 39. *prop.*



given by kind. And ſeeing that the two triangles ABC and DEF have an angle equal to an angle, that is to ſay, the angle BAC to the angle DFE, and the ſides which conſtitute
f 6. 6. thoſe equal angles, proportional; *f* the triangle ABC is alike to the triangle DEF. But the triangle DEF is given by kind: Therefore the triangle ABC is given by kind.

PROP. XLII.



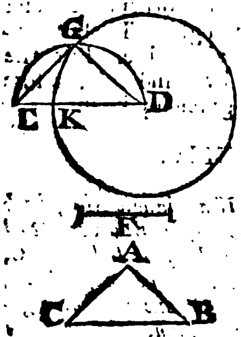
If the ſides of a triangle ABC, be to one another in given reaſons, the triangle ABC is given by kind.

Conſtr. For, Let there be expoſed the right line D, given by magnitude, and ſeeing that the reaſon of BC to AC is given, let the ſame be made of D to E.

Demonſtr. Now D is
a 2. *prop.* given, therefore *a* E is alſo given. Again, ſeeing that the reaſon of AC to AB is given, let the ſame be made of E to F. Now E is
b 2. *prop.* given, therefore *b* F is alſo given. Now of three right lines, equal to the three given right lines D, E, and F, (and of which three lines, two of them, in what manner ſoever they be taken, are greater than the other:) Let there be conſtituted the triangle GHK, in ſuch ſort as D may be equal to HK; but E is equal to KG, and GH equal to F; therefore each of the ſaid lines HK, KG, and GH, is
c 39. *prop.* given by magnitude: Wherefore *c* the triangle HGK is given by kind. And ſeeing that as BC is to CA, ſo is D to E, and that D is equal to HK, and E to KG, as BC is to CA,

ſo HK is to KG. Again, ſeeing that as CA is to AB, ſo is E to F, and that E is equal to KG, and F to GH; as CA is to AB, ſo is KG to GH. But it hath been demonſtrated that as BC is to CA, ſo is HK to KG: Therefore by reaſon of equality, as BC is to AB, ſo is HK to GH. Therefore *d* the triangle d 5. 6. ABC is alſo given by kind.

PROP. XLIII.



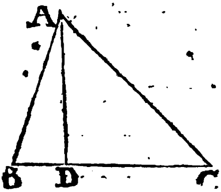
If the ſides BC and BA, about one of the acute angles of a rectangled triangle ABC, have to one another a given reaſon, that triangle is given by kind.

Conſtr. Let there be expoſed the right line DE given by magnitude and poſition, and on it let there be deſcribed the ſemicircle DGE: Therefore *a* the ſemicir- a 6. *def.* cle DGE is given by poſition.

Demonſtr. For the line DE being given, and divided in two equal parts, the center of the ſaid circle is given by poſition, and the ſemidiameter by magnitude. And foraſmuch as the reaſon of BC to BA is given, let the ſame be made of DE to F: Therefore the reaſon of DE to F is given. But DE is given, therefore F *b* is alſo given. Now BC is great- b 2. *prop.* er than *c* AB: Therefore ED is *d* alſo greater c 19. 1. than F. Let DG be fitted equal to F, and let d 14. 5. EG be drawn: then on the center D, with the diſtance DG, let the circle GK be deſcribed. Now that circle *e* is given by poſition, ſeeing e 6. *def.* that the center D is given, and the ſemidia-

meter DG alſo given by magnitude. But the
ſemicircle DGE is alſo given by poſition:
f 25. *prop.* Therefore *f* the point of interſection G is given.
But the points D and E are alſo given, there-
g 26. *prop.* fore *g* each of the right lines DE, DG, and
EG, is given by poſition and magnitude.
h 39. *prop.* Wherefore *h* the triangle DGE is given by
kind. And ſeeing that the triangles ABC and
DGE have an angle equal to an angle, to wit,
i 31. 3. the right angle BAC to the right angle *i* DGE,
and the ſides about the angles CBA and EDG
proportional. But each of the others ACB and
DEG are leſs than a right angle: Thoſe trian-
k 7. 6. gles ABC and DEG *k* are alike. But the
triangle DGE is given by kind: Therefore
the triangle ABC is alſo given by kind.

PROP. XLIV.



If a triangle ABC, hath one angle B given, and that the ſides BA and AC, about another angle BAC, have to one another a given reaſon, the triangle ABC is given by kind.

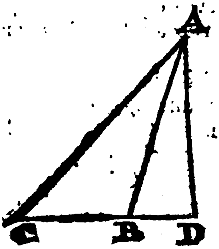
Conſtr. Now the given angle B is either acute or obtuſe, (for it was a right angle in the foregoing Propoſition.) Let it be in the firſt place acute, and from the point A let AD be drawn perpendicular to BC.

Demonſtr. Therefore the angle ADB is gi-
ven: But the angle B is alſo given; and there-
fore the third angle BAD is given: Wherefore
a 40. *prop.* *a* the triangle ABD is given by kind; and
b 3. *def.* therefore *b* the reaſon of BA to AD is given.
But the reaſon of the ſame BA to AC is alſo
c 8. *prop.* given: Therefore *c* the reaſon of AD to AC
is given, and the angle ADC is a right angle:

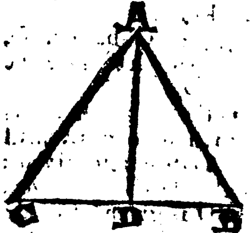
Wherefore the triangle *d* ACD is given by d 43. *prop.*
kind: Therefore *e* the angle C is given. But e 3. *def.*
the angle B is alſo given; and therefore the
other angle BAC is given: Therefore *f* the f 40. *prop.*
triangle ABC is given by kind.

Conſtr. Now let the angle ABC be obtuſe, and on the ſide CB prolonged, let there be drawn the perpendicular AD.

Demonſtr. Foraſmuch as the angle ABC is given, the angle ABD which follows it, ſhall be given. But the angle ADB is alſo given: Therefore the third angle DAB is given.
Wherefore *g* the angle ABD is given by kind; g 40. *prop.*
and therefore *h* the reaſon of DA to AB is given. h 3. *def.*
But the reaſon of AB to AC is alſo given:
Therefore *i* the reaſon of i 8. *prop.*
DA to AC is given, and the angle D is a right angle: Therefore the triangle DAC is given by kind, and therefore the angle ACB is given. But the angle ABC is alſo given: Therefore the third angle BAC is given. Wherefore the triangle ABC is given by kind.



PROP. XLV.



If a triangle ABC, hath one angle BAC given, and that the line compounded of the two ſides AB and AC, about the ſaid given angle BAC, hath to the other ſide BC a given reaſon; the triangle ABC is given by kind.

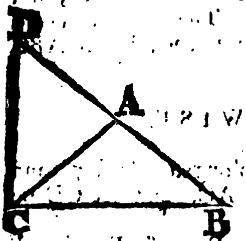
Conſtr. For, let the angle BAC be divided into two equal parts by a 7. *prop.* the line AD, therefore *a* the angle CAD is given.

Demonſtr. Seeing that as AB is to AC, ſo b 3. 6. *b* is BD to CD; by compounding, *c* as the c 18. 5. line compounded of CAB is to CA, ſo is BC to CD, and by permutation, as the line compounded of CAB is to CB, ſo is CA to CD. But the reaſon of the line compounded of CAB to BC is given; therefore the reaſon of CA to CD is alſo given, and the angle CAD is given. Therefore *d* the triangle ACD is given d 44. *prop.* by kind, and therefore the angle C is given. But the angle BAC is alſo given: Therefore, the third angle B is given: Wherefore *e* the e 40. *prop.* triangle ABC is given by kind.

OTHERWISE.

Conſtr. Let BA be prolonged directly unto the point D, in ſuch ſort as that AD may be equal to AC, and let CD be joined.

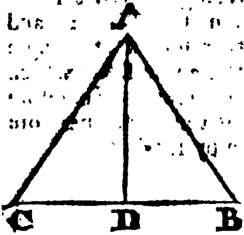
Demonſtr. Foraſmuch as the reaſon of the line compounded of CAB to CB is given, and that AD is equal to AC, the reaſon of the whole line BD to BC is given. But the angle ADC is alſo given, for it is the half of the given angle BAC (for that the ſaid angle BAC [f] is equal to the two internal angles ACD and ADC, which are [g] equal to one another, being the ſides AC and AD are equal:) Wherefore the triangle BDC [h] is given by kind, and therefore the angle B, is given. But the angle BAC is alſo given: Therefore the remaining angle ACB is given: Wherefore [i] the triangle ABC is given by kind.



[f] 32. 1.
[g] 5. 1.
[h] 44. prop.
[i] 40. prop.

PROP. XLVI.

If a triangle ABC hath one angle B given, and that the line CAB compounded of the two ſides, AC and AB, about another angle BAC, hath to the other ſide BC a given reaſon, the triangle ABC is given by kind.



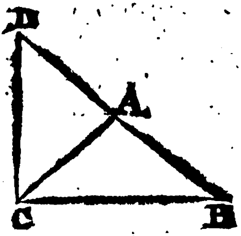
Conſtr. For let the angle BAC be divided into two equal parts, by the line AD.

Demonſtr. Therefore (as hath been ſhewn in the foregoing Propoſition) the compound line CAB is to CB, as AB is to BD. But the reaſon of the ſaid compound line CAB to CB is given: Therefore alſo the reaſon of AB to BD

is given. But the angle B is alſo given:
a 41. prop. Therefore the triangle ABD *a* is given by kind;
b 2. def. and therefore *b* the angle BAD is given. But
the angle BAC is double to that of BAD; and
therefore it is alſo given. Therefore the third
angle C is given. Wherefore the triangle ABC
is given by kind.

OTHERWISE.

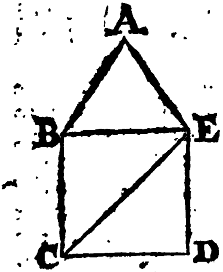
Conſtr. Let BA be prolonged directly, and let AD be put equal to AC, and let CD be joined.



Demonſtr. Foraſmuch as the reaſon of the line compounded of CAB to CB is given, and that AD is equal to AC, the reaſon of BD to BC is given; and the angle B is alſo
c 41. prop. given: Therefore the triangle CDB *c* is given by kind; and
d 3. def. therefore *d* the angle D is given: Therefore the angle BAC which is double to BDC, is alſo given: Wherefore the other angle ACB is given; and therefore the triangle ABC is given by kind.

PROP.

PROP. XLVII.

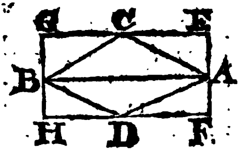


Rectiline figures as ABCDE, given by kind, are divided into triangles given by kind.

Conſtr. For let the right lines EB and EC be drawn.

Demonſtr. Foraſmuch as the rectiline figure ABCDE is given by kind, the angle *a* BAE a 3.def.
is given, and the reaſon of the ſide AB to AE
is alſo given: Therefore *b* the triangle BAE b 41.prop.
is given by kind. Wherefore the angle ABE
is given. But the whole angle ABC is alſo
given: Therefore *c* the remaining angle EBC c 4.prop.
is given. But the reaſon of the ſide AB to
the ſide BE, and alſo that of AB to BC is
given: Therefore *d* the reaſon of BC to BE is d 8.prop.
given, and the angle CBE is alſo given:
Therefore *e* the triangle BCE is given by e 41.prop.
kind. By the ſame diſcourſe it may be demonſtrated that the triangle CDE is given by kind. Therefore the rectiline figures given by kind divide themſelves into triangles given by kind.

PROP. XLVIII.

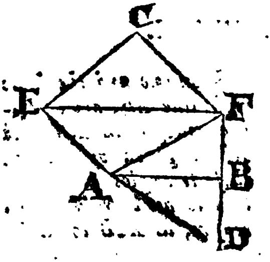


If on one and the ſame right line AB, are deſcribed triangles as ACB and ABD, given by poſition, thoſe triangles ſhall have to one another a given reaſon, as ACB to ABD.

Conſtr. For from the points A and B, let there be drawn at right angles on the line AB, the lines AE and BG, and prolonged unto the points F and H; but by the points C and D, let there be drawn the lines ECG and FDH, parallel to AB.

Demonſtr. Foraſmuch, then as the triangle
a 3. *def.* ABC is given by kind, *a* the reaſon of CA to
BA is given, and the angle CAB alſo given;
but the angle BAE is given; Therefore the
remaining angle CAE is alſo given; but the
angle CAE is given; and therefore the other
b 40. *prop.* angle ACE is alſo given. Wherefore *b* the
triangle AEC is given by kind. Now the rea-
c 8. *prop.* ſon of EA to AB *c* is given; (for *d* the rea-
d 3. *def.* ſon of EA to AC, and that of AC to AB is
given;) and in like manner, the reaſon of FA
e 8. *prop.* to AB is given. Therefore *e* the reaſon of
EA to AF is given; but as AE is to AF, ſo
f 1. 6. *f* the parallelogram AH to the parallelogram
g 41. 1. AG; but ACB is *g* the half of AH, and ADB
the half of AG; therefore the reaſon of the
triangle ACB to the triangle ADB is given;
for it is the ſame reaſon with that of AH
h 15. 5. to AG *h*; that is to ſay, of EA to AF, which
is given.

PROP. XLIX.



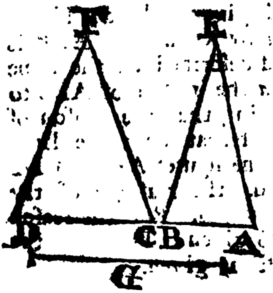
If on one and the ſame right line AB there be deſcribed any two rectiline figures AECFB and ADB, given by kind, they ſhall have to one another a given reaſon (to wit) AECFB to ADB.

Conſtr. For let the lines FA and FE be

drawn: Therefore each of the triangles *a* ABF, a 47. *prop.*
AFE, and ECF is given by kind.

Demonſtr. Seeing that on one and the ſame
right line EF there are deſcribed the triangles
ECF and EAF, given by kind; the reaſon of
ECF to EAF *b* is given. Therefore by com- b 48. *prop.*
pounding, *c* the reaſon of AECF to EAF is c 6. *prop.*
given. But the reaſon of the ſaid EAF to
FAB is given, *d* being they are triangles d 48. *prop.*
given by kind, deſcribed on one and the ſame
right line AF: Therefore *e* the reaſon of AE- e 8. *prop.*
CF to FAB is given. Wherefore by com-
pounding, *f* the reaſon of AECFB to FAB is f 6. *prop.*
given. But the reaſon of the ſame FAB to
ABD *g* is given: Therefore *h* the reaſon of g 48. *prop.*
AECFB to ABD is alſo given. h 8. *prop.*

PROP. L.

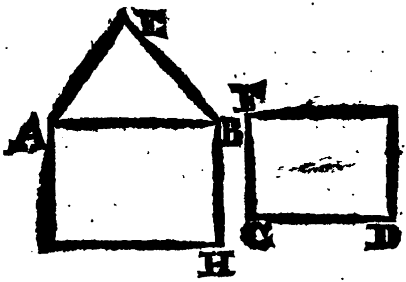


If two right lines AB and CD, have to one another a given reaſon, and that on thoſe lines there be deſcribed rectiline figures AEB and CFD, alike, and alike poſited, they will have to one another a given reaſon.

Demonſtr. To the two lines AB and CD, let there be taken a third proportional: Therefore as AB is to CD, ſo is CD to G. But the reaſon of AB to CD is given: Therefore the reaſon of CD to G is alſo given:
Wherefore *a* the reaſon of AB to G is given. a 8. *prop.*
But *b* as AB is to G, ſo is AEB to CFD: b *cor.* 19.
Therefore the reaſon of the ſame AEB to CFD 20. 6.
is given.

PROP.

PROP. LI.



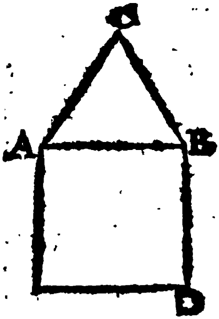
If two right lines AB and CD, have to one another a given reaſon, and that upon them there be deſcribed any rectiline figures AEB and CFD, given by kind, they will have to one another a given reaſon, (to wit, that of AEB to CFD.)

Conſtr. For on AB let the rectangled figure AH be deſcribed alike and alike poſited to DF.

Demonſtr. Now DF is given by kind: Therefore alſo AH is given by kind. But AEB is alſo given by kind, and deſcribed on the ſame
a 49. *prop.* line AB: Therefore *a* the reaſon of AEB to AH is given: And ſeeing that the reaſon of AB to CD is given, and that on thoſe lines are deſcribed the rectiline figures AH and DF
b 50. *prop.* alike, and alike poſited, the reaſon *b* of the ſaid line AH to DF is given. But the reaſon of AEB to AH is alſo given: Therefore the
c 8. *prop.* reaſon *c* of AEB to DF is given.

PROP.

PROP. LII.



If on a right line AB, given by magnitude, there be deſcribed a figure ACB, given by kind, that figure ACB is given by magnitude.

Conſtr. For on the ſame line AB, let the ſquare AD be deſcribed. Therefore AD is given by kind * and by magnitude.

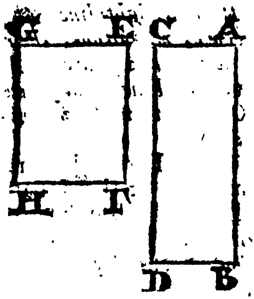
Demonſtr. Seeing that on the right line AB, are deſcribed the two rectiline figures ACB and AD, given by kind, *a* the reaſon of ACB to AD is given: Therefore *b* ACB is given by magnitude. a 39. *prop.* b 2. *prop.*

Scholium.

* *The antient Interpreter hath noted here that every ſquare is given by kind; for that all the angles thereof are given; being all equal and right angles: But alſo the reaſons of the ſides are given; for thoſe ſides being all equal, their reaſons are alſo equal. Moreover, whenſoever a ſquare is expoſed, a ſquare equal thereto may be exhibited; and therefore the ſquare is given by magnitude, as alſo each ſide thereof.*

PROP.

PROP. LIII.

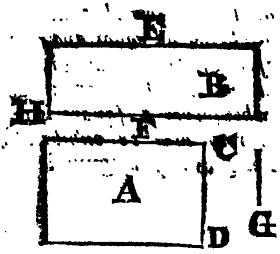


If there be two figures AD and EH, given by kind, and that one ſide BD, of the one, hath to a ſide FH of the other, a given reaſon; the other ſides ſhall have alſo to the other ſides given reaſons.

Demonſtr. For ſeeing that the reaſon of BD to FH is given, and alſo that *a* of BD to BA, *b* the reaſon of the ſaid AB to FH is given. But the reaſon of the ſame FH to FE *c* is alſo given: Therefore *d* the reaſon of AB to EF is given. In like manner alſo the reaſons of the other ſides to the other ſides are given.

a 3. *def.*
b 8. *prop.*
c 3. *def.*
d 8. *prop.*

PROP. LIV.



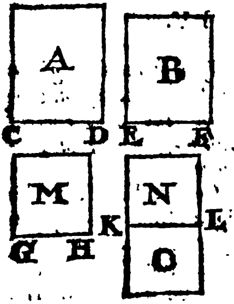
If two figures A and B, given by kind, have to one another a given reaſon; alſo their ſides ſhall be to one another in a given reaſon.

Conſtr. For either the figure A is alike and alike poſited to B, or is not: Let it in the firſt place be alike, and alike poſited; and let there be taken the line G, a third proportional to the lines CD and EF.

Demonſtr. As CD is to G, *a* ſo is A to B. a *cor.* 19,
But the reaſon of A to B is given; therefore 20. 6.
alſo the reaſon of CD to G is given. And
ſeeing that CD, EF, and G, are proportional,
b alſo the reaſon of CD to EF is given. But b 24. *prop.*
A and B are given by kind: Therefore *c* the c 53. *prop.*
other ſides ſhall have given reaſons to the
other ſides.

Now let the figure A be not alike to the figure B, and let there be deſcribed on EF the figure EH, alike and alike poſited to A: Therefore the figure EH is given by kind; but the figure B is alſo given by kind: Therefore *d* the reaſon of B to EH is given; and d 49. *prop.*
therefore the reaſon of A to the ſame EH *e* is e 8. *prop.*
alſo given: But A is alike to EH: Therefore (by what is aboveſaid) the reaſon of CD to EF is given; and in like manner the reaſon of the other ſides to the other ſides is given.

OTHERWISE.



Conſtr. Let there be expoſed the given line GH: Now either the figure A is alike to the figure B, or not. Let it in the firſt place be alike, and let it be as CD is to EF, ſo is GH to LK; then on GH and LK let the figures M and N be deſcribed alike, and alike poſited to the ſaid A and B, which figures M and N ſhall be conſequently given by kind.

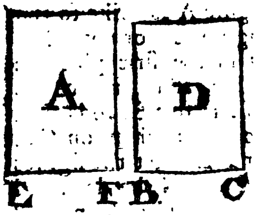
Demonſtr. Therefore ſeeing that as CD is to EF, ſo is GH to LK, and that on thoſe lines CD, EF, GH, and LK, are deſcribed the figures

f 22. 6. A, B, M, and N, alike and alike poſited; *f* as A is to B, ſo is M to N. But the reaſon of A to B is given: Therefore the reaſon of M
g 52. *prop.* to N is given, But *g* M is given, conſidering that it is a figure given by kind, deſcribed on a right line given by magnitude; therefore N is alſo given.

Conſtr. 2. Now, on LK let the ſquare O be
h *ſch.* 52. *prop.* deſcribed: Therefore *h* the figure O is given by kind.

Demonſtr. 2. Wherefore the reaſon of O to N is given. But N is given: Therefore O
i *ſch.* 52. *prop.* is given; and conſequently, *i* alſo KL. But GH is given: Therefore *k* the reaſon of GH to
k 3. *prop.* KL is given. But as GH is to LK, ſo is CD to EF. Therefore the reaſon of CD to EF is given; and therefore the figures A and B being
l 53. *prop.* given by kind, *l* the other ſides of the ſame figures ſhall alſo have to the other ſides given reaſons. But if the figures be not alike, the latter part of the demonſtration here above muſt be obſerved.

PROP. LV.



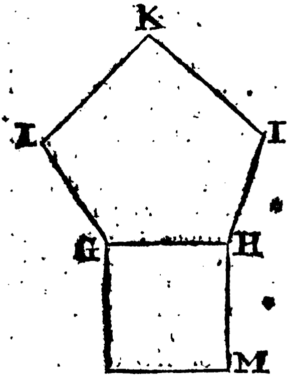
If a ſpace A be given by kind, and by magnitude, the ſides thereof ſhall be given by magnitude.

Conſtr. For, let the right line BC given by poſition and by magnitude, be expoſed; and thereon let there be deſcribed the ſpace D, alike and alike poſited to A; therefore the ſaid ſpace D is given by kind.

Demonſtr. For that it is deſcribed on the line BC, given by magnitude, it is alſo *a* given
a 52. *prop.* by magnitude. But the figure A is alſo given: There-

Therefore *b* the reaſon of A to D is given. b 3. *prop.*
But thoſe figures A and D are given by kind:
Therefore *c* the reaſon of the line EF to the c 54. *prop.*
line BC is given. But BC is given: Therefore
d EF is alſo given. But the reaſon of the d 3. *def.*
ſame EF to FG is given: Therefore *e* FG is e 2. *prop.*
given. And by the ſame reaſons it may be demonſtrated that each of the other ſides are given by magnitude.

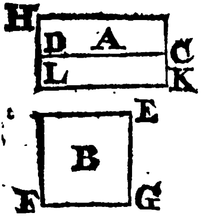
OTHERWISE.



Conſtr. Let the ſpace GHIKL be given by kind and by magnitude: I ſay that the ſides thereof are given by magnitude. For on the right line GH let there be deſcribed the ſquare GM; therefore *f* f *Sch.* 52.
GM is given by *prop.*
kind.

Demonſtr. But the ſpace GHIKL is alſo given by kind:
Therefore *g* the reaſon of the ſame ſpace GK g 49. *prop.*
to GM is given. But GK is given by magnitude: Therefore *h* GM is alſo given by magni- h 2. *prop.*
tude; and ſeeing that GM is the ſquare of the
line GH, *i* that line GH is given by magni- i *Sch.* 52.
tude. Wherefore in like manner, each of the *prop.*
other lines HI, IK, KL, and LG, is given.

PROP. LVI.



If two equiangled parallelograms A and B, have to one another a given reaſon; as one ſide CD of the firſt A, is to one ſide FG, of the ſecond B; ſo the other ſide GE, of the ſecond B, is to that to which DH the other ſide of the firſt A, hath the given reaſon that the parallelogram A hath to the parallelogram B.

Conſtr. For let HD be prolonged directly to L, ſo that as CD is to FG, ſo HD may be to DL; and finiſh the parallelogram DK.

Demonſtr. Seeing that as CD is to FG, ſo
a 34. 1. HD is to DL, and *a* that CD is equal to KL; as LK is to FG, ſo is GE to DL; and thus the ſides about the equal angles DLK and EGF are reciprocally proportional: Wherefore
b 14. 6. *b* DK is equal to B; and therefore ſeeing the reaſon of A to B is given, and that B is equal to DK, the reaſon of A to DK is given. But
c 1. 6. as *c* A is to DK (that is to B) ſo is HD to DL; therefore the reaſon of HD to DL is alſo given: and ſeeing that as CD is to FG, ſo GE is to DL, and that the right line HD hath to DL a given reaſon; to wit, that which the ſpace A hath to the ſpace B; as CD is to FG, ſo GE is to that to which HD hath the given reaſon that the ſpace A hath to the ſpace B, that is to ſay, the reaſon of HD to DL.

PROP.

PROP. LVII.

If a given ſpace AD be applied to a given right line AB in a given angle CAB, the breadth CA of the application is given.

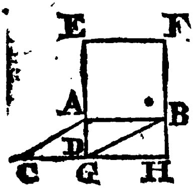
Conſtr. For on AB, let there be deſcribed the ſquare AF; therefore *a* the ſame AF is given: Let the lines EA, FB, and CD, be prolonged to the points G and H. a ſch. 52. prop.

Demonſtr. Seeing therefore that each ſpace AD and AF is given, their reaſon is alſo given. But *b* AD is equal to AH: Therefore the reaſon of AF to AH is given: Wherefore the reaſon of EA to AG is given. (for *c* it is the ſame with that of AF to AH.) But EA is equal to AB; therefore the reaſon of AB to AG is given. Now ſeeing that the angle CAB is given, and the angle GAB alſo given, the reſidue CAG is given. But the angle CGA is alſo given, being a right angle: Therefore the remaining angle ACG is given. Wherefore the triangle *d* CAG is given by kind. Therefore the reaſon of CA to AG is given. But the reaſon of AB to the ſame AG is alſo given: Therefore the reaſon of CA to AB is given; and the ſaid AB is given: Wherefore CA is alſo given.

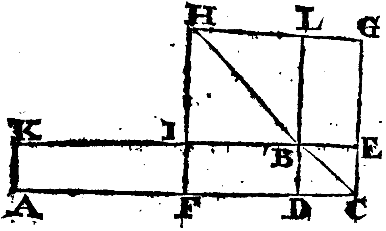
b 36. 1.

c 1. 6.

d 40. prop.



PROP. LVIII.



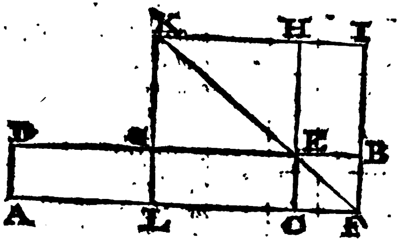
If a given ſpace AB, be applied to a given right line AC, wanting by a figure DE, given by kind, the breadths of the defects are given.

Conſtr. For let AC be divided in two equal parts in the point F: Therefore as well AF as FC is given. On the ſaid line FC let there be deſcribed the rectangled figure FG alike and alike poſited to DE. Therefore FG is given by kind.

Demonſtr. Seeing the figure FG is deſcribed
on the right line FC given by magnitude, the
a 52. *prop.* ſaid rectiline FG is *a* alſo given by magnitude.
b 36. 1. But FG is equal to AB and IL; (for *b* AI
c 43. 1. and FE being equal, and *c* FB and BG alſo
equal, the Gnomon ICL is equal to AB; and
therefore their added figure IL, common to
both, FG ſhall be equal to AB and IL:)
Therefore the figures AB and IL together are
given by magnitude. But AB is given by
d 4. *prop.* magnitude: Therefore *d* the remaining figure
IL is alſo given by magnitude. But it is alſo
e 24. 6. given by kind, ſeeing it is *e* alike to DE:
f 55. *prop.* Therefore *f* the ſides of the ſame IL are
given: Wherefore IB is given; and ſeeing
g 34. 1. that it is equal *g* to FD, the ſame FD is al-
h 4. *prop.* ſo given. But FC is given, therefore the re-
i 3. *def.* mainder DC *h* is given; and *i* in a given
k 2. *prop.* reaſon to BD, and therefore *k* BD is given.

PROP.

PROP. LIX.

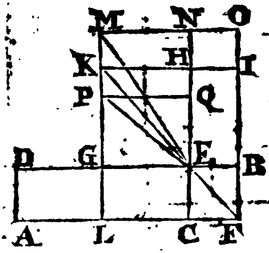


If a given ſpace AB be applied according to a given right line AC, exceeding it by a figure CB given by kind, the breadths of the exceſſes CE and CF are given.

Conſtr. For DE being divided into two equal parts in G, let there be deſcribed on GE the rectiline figure GH, alike and alike poſited to CB.

Demonſtr. Now ſeeing that CB is alike to GH, thoſe figures CB and GH * are about one and the ſame diameter, and GH is given by kind, as is CB. But it is deſcribed on the given line GE: Therefore *a* the ſame GH is a 32. *prop.*
alſo given by magnitude. But AB is given: Therefore AB and GH are given by magnitude. Now thoſe figures AB and GH, are equal to LI, (for AC, LE, and EI, being equal, the Gnomon GFH is equal to AB; and therefore adding GH common to both, LI ſhall be equal to AB and GH;) therefore LI is given by magnitude; but it is alſo given by kind, being it is *b* alike to CB. Therefore b 24. 6.
c the ſides of the ſaid LI are given, ſeeing c 55 *prop.*
it is equal to GE: Therefore *d* the remainder d 4. *prop.*
CF is given, and in a given reaſon *e* to CE. e 3 *def.*
Wherefore *f* CE is given. f 2 *prop.*

Scholium.

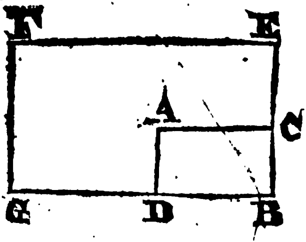


** EUCLIDE ſuppoſeth here that CB and GH are about one and the ſame diameter, but we ſhall thus demonſtrate it: Let CB and GH be two alike parallelograms diſpoſed as above, that is to ſay, that the equal angles join together in E, the ſide CE meets directly with his homologal ſide EH, and the ſide BE, his correſpondent ſide EG; and let the diameter FE be drawn, I ſay that the ſaid diameter FE prolonged, will paſs by the point K; that is to ſay, the parallelograms GH and CB, conſiſt about one and the ſame diameter. For if it be denied, the diameter EF being produced, will paſs above the point K, or below it. Let it in the firſt place paſs above it, and let it cut GK, prolonged in the point M, and by the point M, let there be drawn MN, parallel to KH, which ſhall meet EH, prolonged in the point N, and FB in O.*

Demonſtr. Foraſmuch as the parallelograms
GN and CB are with the parallelogram LO
g 24. 6. about one and the ſame diameter, they are g
alike to one another. Wherefore as FC is to
CE, ſo is EG to GM. In like manner, ſeeing
the parallelograms CB and GH are alike, as
h 11. 5. FC is to CE, ſo is EG to GK: Therefore h
as EG is to GM, ſo is EG to GK. Where-
i 9. 5. fore i GM and GK are equal, a part to the
whole, which is abſurd: By the ſame rea-
ſons it may be demonſtrated; that the diame-
ter prolonged will not fall below the point K:

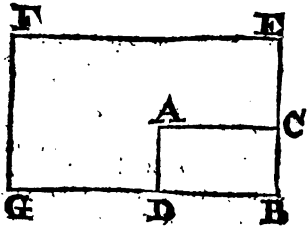
Therefore the parallelograms CB and GE conſiſt about one and the ſame diameter.

PROP. LX.



If a parallelogram AB, given by kind and by magnitude, be augmented or diminiſhed by a Gnomon CFD, the breadths of the Gnomon (conſiſting of the lines CE and DG) are given.

Demonſtr. For ſeeing that AB is given, and the Gnomon CFD alſo given, the whole parallelogram BF is given: But it is alſo given by kind, ſeeing it is alike to BA: Therefore *a* the ſides of the ſame BF are given; and therefore each of the lines BE and BG is given. But each of the lines BC and BD is given; therefore each of the remaining lines CE and DG is alſo given. a 55. prop.

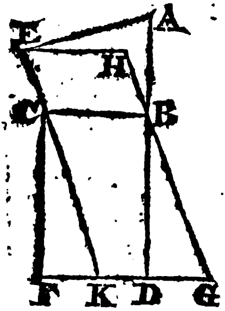


Conſtr. Now let the parallelogram BF, given by kind and by magnitude, be diminiſhed by the given Gnomon CFD: I ſay that each of the lines CE and DG is given.

Demonſtr. 2. For ſeeing that BF is given, and the Gnomon CFD given, the remaining figure AB is alſo given. But it is alſo given

by kind, ſeeing it is alike to BF: Therefore
b 55.*prop.* *b* the ſides of the ſaid AB are given, and therefore each of the lines CB and BD is given. But each of the lines BE and BG is given: Therefore alſo each of the remaining lines CE and DG is given.

PROP. LXI.



If to one ſide of a figure ABCE, given by kind, there be applied a ſpace parallelogram CD, in a given angle BCF, and that the given figure AC hath to the parallelogram CD a given reaſon, the parallelogram CD is given by kind.

Conſtr. For by the point B, let BH be drawn parallel to CE, and by the point E let EH be drawn parallel to CB, and let EC and HB be prolonged to the points K and G.

Demonſtr. Foraſmuch as the angle BCE is
a 3.*def.* given, and the reaſon of EC to CB, *a* the parallelogram CH is given * by kind. But the figure ABCE is alſo given by kind, and is deſcribed on the ſame line BC, as the parallelogram CH given by kind is: Therefore *b*
b 49.*prop* the reaſon of the figure ABCE to the parallelogram CH is given. But by ſuppoſition, the reaſon of the ſaid figure ABCE to the parallelogram CD is alſo given; and CD is *c*
c 36. 1. d 8.*prop.* equal to CG: Therefore *d* the reaſon of CH to CG is given. Wherefore the reaſon of
e 1.6. the line EC to the line CK is given; (for *e* as CH is to CG, ſo is EC to CK.) But the reaſon of EC to CB is alſo given: Therefore
f 8.*prop.* *f* the reaſon of the ſaid CB to CK is given.

And ſeeing that the angle ECB is given, alſo
the following angle BCK *g* is given. But the g 13.1.
angle BCF is propoſed given; and therefore 4.*prop.*
the remaining angle FCK is given. Alſo the
angle CKF is given, for that *h* it is equal h 29. 1.
to the angle BCK: Therefore the other angle
CFK is given: Wherefore *i* the triangle FCK i 40. *prop.*
is given by kind; and therefore the reaſon
of FC to CK is given. But the reaſon of CB
to the ſame CK is alſo given: Therefore *k* k 8. *prop.*
the reaſon of FC to CB is given; and the
angle BCF is alſo given. Wherefore the parallelogram CD is given by kind.

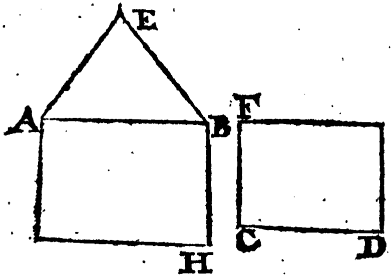
Scholium.

* *Altho' it be manifeſt that a parallelogram that hath one angle given, and the reaſon of the ſides about the ſame angle alſo given, is given by kind, as* Euclide *doth here declare, ſo it is notwithſtanding that the antient Interpreter doth thus demonſtrate it.*

Seeing that in the parallelogram CH the angle ECB is given, the angle CEH is alſo given; for the right line EC falling on the parallels EH and CB, doth make the two internal angles on the ſame part equal to two right angles. And therefore ſeeing that the angle ECB is given, the other angles are given; and ſeeing that the reaſon of EC to CB is given, and that BH is equal to CB, and EH to BC, the reaſon of the ſides to one another is alſo given.

PROP.

PROP. LXII.

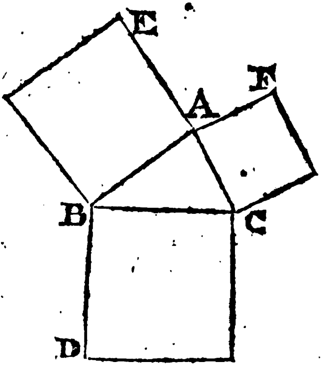


If two right lines AB and CD, have to one another a given reaſon, and that on one of them AB, there be deſcribed a figure AEB, given by kind; but on the other CD, a ſpace parallelogram DF, in a given angle DCF, and that the figure AEB hath to the parallelogram DF a given reaſon, the parallelogram DF is given by kind.

Conſtr. For on the line AB let there be deſcribed the parallelogram AH, alike and alike poſited to DF.

Demonſtr. Seeing then that the reaſon of AB
to CD is given, and that on thoſe lines are
deſcribed the rectiline figures AH and FD,
a 50. *prop.* alike and alike poſited, *a* the reaſon of AH
to FD is given. But the reaſon of FD to AEB
b 8. *prop.* is alſo given: Therefore *b* the reaſon of AH
to AEB is given. But the angle ABH is al-
ſo given, being equal to the angle FCD; and
ſo the figure AEB is given by kind; and to
AB one of the ſides thereof, the parallelogram
AH is applied in a given angle ABH, and
c 61. *prop.* the reaſon of the ſaid figure AEB to the ſaid
parallelogram AH is given: Therefore *c* the
parallelogram AH is given by kind; and
therefore FD which is alike thereto, is alſo
given by kind.

PROP. LXIII.

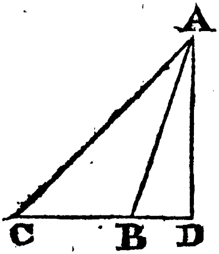


If a triangle ABC be given by kind, the ſquare BE, CD, and CF, which is deſcribed on each of the ſides, ſhall have a given reaſon to the triangle ABC.

Demonſtr. For ſeeing that on one and the ſame right line BC, there are deſcribed the two rectiline figures ABC and CD, given by kind, *a* the reaſon of the ſame ABC to CD is given; and therefore the reaſon of the ſquares BE and CF, to the triangle ABC, is alſo given.

a 49. prop.

PROP. LXIV.



If a triangle ABC, hath an obtuſe angle ABC given, that ſpace of which the ſide AC ſubtending the obtuſe angle ABC, is more in power than the ſides AB and BC, that comprehend the ſaid angle, ſhall have a given reaſon to the triangle ABC.

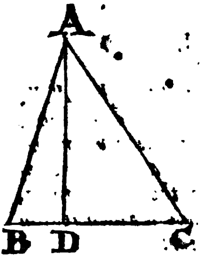
Conſtr. Let the line CB be prolonged directly, and from the point A let the perpendicular AD be drawn: I

ſay that the ſpace of which the ſquare of the
line AC doth exceed the ſquares of the lines
a 12. 2. AB and BC, that is to ſay, *a* the double of
the rectangle contained under CB and BD,
ſhall have a given reaſon to the triangle ABC.

Demonſtr. Forſeeing that the angle ABC is
given, the angle ABD is alſo given. But the
angle ADB is alſo given; therefore the other
b 40. *prop.* angle BAD is given: Wherefore *b* the triangle
c 3. *def.* ABD is given by Kind; therefore *c* the reaſon
of AD to DB is given. But as AD to DB, ſo
d 1. 6. *d* the rectangle of AD and BC is to the rectangle
of BC and BD. But the reaſon of AD to
BD is given: Therefore alſo is the reaſon of
the rectangle of AD and BC to the rectangle
of BC and BD given: Wherefore the reaſon
of the double of the ſaid rectangle BC and
BD to the rectangle of AD and BC is alſo
given. But the ſaid rectangle of AD and
BC hath alſo a given reaſon to the triangle
ABC (to wit, double reaſon; for the rectangle
e 41. 1. is *e* double to the triangle) therefore the reaſon
of the double of the rectangle of BC and
f 8. *prop.* BD *f* to the triangle ABC is given. But the
ſame double of the rectangle of CB and BD
is that ſpace of which the ſquare of the
line AC doth exceed the ſquares of the lines
AB and BC; Therefore the ſame ſpace hath
a given reaſon to the triangle ABC.

PROP.

PROP. LXV.



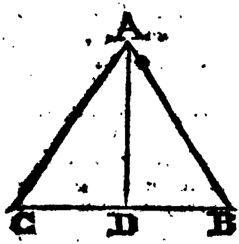
If a triangle ABC, hath one acute angle ACB given, that ſpace, of which the ſide ſubtending the ſaid acute angle is leſs in power than the ſides comprehending the ſame acute angle, ſhall have a given reaſon to the triangle.

Conſtr. From the point A let there be drawn the
line AD, perpendicular to BC: I ſay that ſpace
of which the ſquare of the line AB is leſs
than the ſquares of the lines AC and CB,
that is to ſay, *a* the double of the rectangle a 13. 2.
of BC and CD, hath a given reaſon to the
triangle ABC.

Demonſtr. For ſeeing that the angle C is
given, and the angle ADC alſo given, the
other angle DAC is given: Wherefore the
triangle *b* ADC is given by kind; and therefore b 40. *prop.*
the reaſon of AD to DC is given, and
conſequently alſo *c* that of the rectangle of c 1. 6.
BC and CD to the rectangle of BC and AD:
Therefore the reaſon of the double of the rectangle
of BC and CD to the rectangle of BC
and AD is given. But the reaſon of the ſame
rectangle of BC and AD to the triangle ABC
is given, (for *d* the rectangle is double to the
triangle:) Therefore *e* the reaſon of the double d 41. 1.
of the rectangle of BC and CD to the e 8. *prop.*
triangle ABC is given. And ſeeing that the
ſame double of the rectangle of BC and CD
is that whereof the ſquare of the line AB is
leſs than the ſquares of the lines AC and BC,
that ſpace of which the ſquare of the line
AB

AB is leſs than the ſquares of the lines AC and BC, ſhall have a given reaſon to the triangle ABC.

PROP. LXVI.



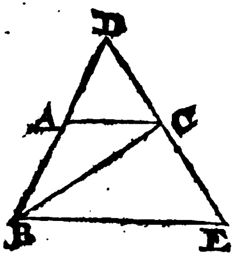
If a triangle ACB, hath one angle B given, the rectangle made of the lines AB and BC, containing the ſame angle, ſhall have a given reaſon to the triangle.

Conſtr. For from the point A let AD be drawn perpendicular to CB.

Demonſtr. Therefore ſeeing that the angle B is given, and alſo the angle ADB; the other angle BAD is likewiſe given. Wherefore the
a 40. *prop.* triangle ADB *a* is given by kind; and conſequently the reaſon of AB to AD is given.
b 1. 6. But as AB is to AD, *b* ſo the rectangle of AC and CB is to the rectangle of CB and AD: Therefore the reaſon of the rectangle of AC and CB to the rectangle of CB and AD is given. But the reaſon of the ſaid rectangle of CB and AD to the triangle ACB is alſo given; (for that it is double reaſon, the
c 41. *prop.* rectangle being double *c* to the triangle:)
d 8. *prop.* Therefore *d* the reaſon of the rectangle of AC and CB to the triangle ABD is given.

PROP

PROP. LXVII.



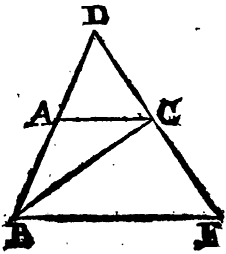
If a triangle ABC hath one angle BAC given, that ſpace by which the ſquare of the line compounded of the two ſides BA and AC, that contain the ſame given angle BAC doth exceed the ſquare of the other ſide, it ſhall have a given reaſon to the triangle ABC.

Conſtr. For let BA be prolonged in ſuch ſort as that AD may be equal to AC, then having drawn DCE infinitely, from the point B let BE be drawn parallel to AC, meeting the ſaid DE in the point E.

Demonſtr. Foraſmuch as AD is equal to AC, *a* DB is equal to BE; (for the two triangles ADC and BDE are alike) and from the top B is drawn to the baſe DE, the right line BC: Therefore * the rectangle of DC and CE, with the ſquare of BC, is equal to the ſquare of BD; but the ſame BD is compounded of BA and AC; therefore the ſquare of the compound of AB and AC is greater than the ſquare of BC, of the rectangle of DC and CE. a 4. 6. & 14. 5.

Now I ſay that the rectangle of DC and CE hath a given reaſon to the triangle ABC: Foraſmuch as the angle BAC is given, the angle DAC is alſo given. But each of the angles ADC and ACD is given, it being the half of the angle BAC which is given. Therefore *b* the triangle ADC is given by kind; and therefore the reaſon of DA to DC is given. Therefore *c* the reaſon of the ſquare of the ſaid DA to the ſquare of DC is alſo

b 40. *prop.*
c 50. *prop.*



d 2. 6. given. And ſeeing that as BA is to AD, *d*
ſo is EC to CD, and
alſo as BA is to AD,
e 1. 6. *e* ſo is the rectangle of
BA and AD to the
ſquare of AD; and as
f 1. 6. EC is to CD, *f* ſo alſo
is the rectangle of EC
and CD to the ſquare
of CD; by permutation,
as the rectangle of BA
and AD is to the rectangle
of EC and CD,
ſo is the ſquare of AD to the ſquare of DC.
But the reaſon of the ſaid ſquare of AD to
the ſquare of DC is given: Therefore the reaſon
of the rectangle of BA and AD to the
rectangle of EC and CD is alſo given. But
AD is equal to AC: Therefore the reaſon of
the rectangle of BA and AC to the rectangle
of EC and CD is given. But the reaſon of
the rectangle of BA and AC to the triangle
g 66. *prop.* ABC *g* is given, becauſe the angle BAC is
h 8. *prop.* given: Therefore *h* the reaſon of the rectangle
EC and CD to the triangle ABC is given.
But the rectangle of EC and CD is that
whereof the ſquare of the line compounded of
BA and AC is greater than the ſquare of BC:
Therefore that ſpace to which the ſquare of
the line compounded of BA and AC is greater
than the ſquare of BC, ſhall have a given
reaſon to the triangle ABC.

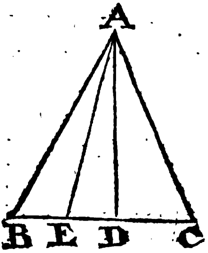
Scholium.

* *EUCLIDE ſuppoſeth in this place that when in an Iſoſceles triangle a right line is drawn from the top to the baſe, the ſquare of that line, with the rectangle contain'd under the ſegments of the*

the baſes, is equal to the ſquare of either of the other legs, which the antient interpreter doth thus demonſtrate.

Conſtr. Let ABC be an Iſoſceles triangle, whoſe legs are AB and AC; and from the top A let AD be drawn to the baſe BC: I ſay that the ſquare of AD with the rectangle of BD and DC, is equal to the ſquare of either of the legs AB or AC.

Demonſtr. Now the line AD is perpendicular to BD, or not: Let it in the firſt place be perpendicular: Therefore it will cut the baſe BC into two equal parts in the point D; and therefore the rectangle contained under BD and DC is equal to the ſquare of the ſaid BD, and adding to them the common ſquare of AD, the rectangle of BD and DC with the ſquare of AD, ſhall be equal to the ſquares of DB and AD: But to thoſe ſquares of AD and DB *i* the ſquare of AB is equal: Therefore the ſquare of AB is equal to the rectangle of BD and DC, and the ſquare of AD together. i 47. 1.



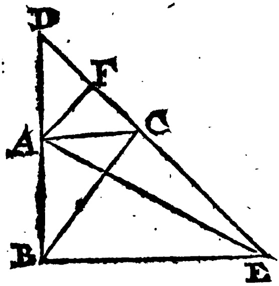
Now ſuppoſe AD not to be perpendicular, but that from the point A there doth fall on BC the perpendicular AE, that being ſo, BC ſhall be cut into two parts equally in the point E, and unequally in D. Wherefore the rectangle of BD and DC, with the ſquare of DE, *k* is equal to the ſquare of BE; and adding the common ſquare of AE, the rectangle of BD and DC, with the ſquares of DE and AE, ſhall be equal to the ſquares of BE and AE. But *l* the ſquare of AD is equal to the two ſquares of DE and AE: Therefore k 5. 2. l 47. 1.

the rectangle of BD and DC, with the ſquare of AD is equal to the ſquares of BE and AE. But to theſe ſquares of BE and AE the ſquare of AB is equal: Therefore the ſquare of AD, with the rectangle of BD and DC, is equal to the ſquare of AB.

OTHERWISE.

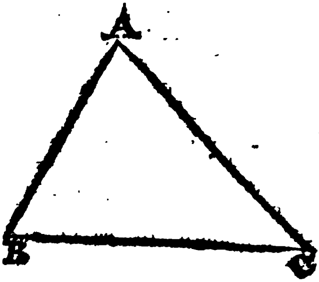
Conſtr. Having done, as in the foregoing Demonſtration, from the point A, let AF be drawn perpendicular to CD, and let AE be drawn.

Demonſtr. Foraſmuch as the angle BAC is given, the half thereof ACF ſhall be alſo given. But the angle AFC is given; and therefore the triangle AFC is given by kind: Therefore the reaſon of AF to FC is given. But the reaſon of CD to the ſame FC is alſo given, ſeeing that CD is double to FC:
m 8. *prop.* Therefore *m* the reaſon of CD to AF is given; and therefore alſo the reaſon of the rectangle of CD and EC, to the rectangle of AF and
n 1. 6. EC, is given; (for it is the ſame reaſon *n* as that of CD to AF.) But the reaſon of the rectangle of AF and FC to the triangle ACE is given; ſeeing it is
o 41. 1. double *o* to the ſame triangle. Therefore the reaſon of the rectangle of CD and CE to the triangle ACE is alſo given. But the triangle ACE is equal to the triangle
p 37. 1. ABC *p*, they being both conſtituted on one and the ſame baſe AC, and between the ſame parallels



parallels AC and BE: Therefore *q* the reaſon q 8. *prop.*
of the rectangle of CE and CD to the triangle ABC is given. But the ſaid rectangle of CE and CD is the ſpace by which the ſquare of the line compounded of AB and AC, is greater than the ſquare of BC: Therefore that ſpace by which the ſquare of the line compounded of AB and AC is greater than the ſquare of BC, hath a given reaſon to the triangle ABC.

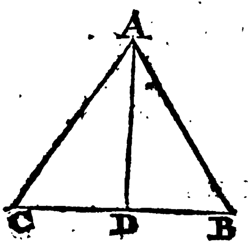
OTHERWISE.



For the given angle A is either a right, acute, or obtuſe angle: Let it in the firſt place be ſuppoſed a right angle: Therefore the ſquare of the line compounded of BAC, is greater than the ſquare of BC, by twice the
rectangle of BA and AC; (ſeeing that *r* the r 47. 1.
ſquare of BC is equal to the ſquares of BA and AC; and the ſquare of the line com-
pounded of BAC *s* is equal to thoſe two s 5. 2.
ſquares of BA and AC, and twice the rectangle of the ſaid BA and AC:) Wherefore the reaſon of double the rectangle of BA and AC to the triangle ABC is given.

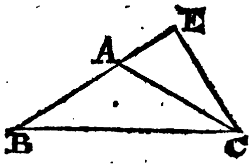
Conſtr. Now let the angle C be ſuppoſed acute, and from the point A let there be drawn on CB the perpendicular AD.

Demonſtr. Foraſmuch as the triangle CAB is an Oxigonium triangle, and the perpendicular AD being drawn, the ſquare of CA and CB



t 13. 5. are equal *t* to the ſquare of AB with twice the rectangle of CB and CD; adding therefore the common double rectangle of CA and CB, the ſquares of CA and CB, with the double rectangle of the ſaid CA and u 4. 2. CB, that is to ſay, *u* the alone ſquare of the line compounded of ACB, are equal to the ſquare of AB, with the double of the rectangle of CD and CB, and over and above the double of the rectangle of AC and CB, that is to ſay, the double of the rectangle contained under the compound line of ACD and CB (for x 1. 2. the rectangle of ACD and CB is *x* equal to the rectangles of AC and CB, and of CD and CB:) Therefore the ſquare of the line compounded of ACB is greater than the ſquare of AC, by double the rectangle of ACD and CB. And ſeeing that the angle ACB is given, and the angle BDA alſo given, the other y 40 *prop.* angle CAD is given: Therefore *y* the triangle CAD is given by kind, and therefore the reaſon of CD to CA is given, and by conſequence the reaſon of the line compounded z 6. *prop.* of ACD to CA *z* is alſo given. Wherefore the reaſon of the rectangle of thoſe lines a 1. 6. compounded of ACD and CB *a* to the rectangle of AC and CB is alſo given. But the reaſon of the ſaid rectangle of AC and CB b 66. *prop.* to the triangle CAB *b* is given, ſeeing the angle C is given; therefore the reaſon of double the rectangle of the line compounded of ACD and CB to the triangle CAB is given.

Laſtly,



Laſtly, Let the angle BAC be ſuppoſed to be obtuſe, and having prolonged BA from the point C, let the perpendicular CE be drawn on the ſaid line BA prolonged; and let AF be propoſed to be equal to AE.

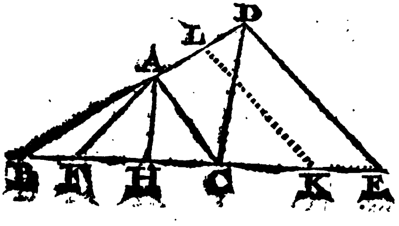
Demonſtr. Foraſmuch as the angle BAC is obtuſe, and the perpendicular CE being drawn, the ſquares of AB and AC, and the double of the rectangle under BA and AE, or AF, are all alike equal *c* to the ſquare of BC, and c 12. 2.
adding the common double rectangle of BA and AC, the ſquares of the ſaid AB and AC, with the double of the rectangle of the ſame AB and AC, that is to ſay, *d* the ſquare of d 4. 2.
the line compounded of BAC, and the double of the rectangle of BA and AF are together equal to the ſquare of BC, with the double of the rectangle of BA and AC. Let the common double of the rectangle of BA and AF be taken away, and there will remain the ſquare of the line compounded of BAC, equal to the ſquare of BC, with the rectangle of AB and CF; (for the rectangle of AB and AC is equal *e* to the two rectangles of e 1. 2.
AB and AE, and of AB and CF:) Therefore the ſquare of the line compounded of BAC is greater than the ſquare of BC by the double of the rectangle of AB and CF. And foraſmuch as the angle BAC is given, the angle CAE *f* is given. But the angle AEC is f 13. 11.
alſo given; therefore the other angle ACE is given: Wherefore *g* the triangle ACE is given g 40. *prop.*
by kind, and therefore the reaſon of CA to AE, that is to ſay, to AF is given. Therefore *h* the reaſon of the ſaid CA to FG is h 5. *prop.*

alſo given. But the reaſon of the ſame CA
i 8. *prop.* to CE is given; therefore *i* the reaſon of CE
to CF is alſo given. Wherefore the reaſon of
the rectangle of EC and AB to the rectangle
of FC and AB is given; (for the rectangle
k 2. 6. gle is to the rectangle *k* as CE is to CF) and
alſo that of the rectangle of AC and AB,
l 8. *prop.* to the rectangle of EC and AB. Therefore *l*
the reaſon of the rectangle of FC and AB to
the rectangle of AC and AB is given. But
the reaſon of the rectangle of AC and AB
m 66. *prop.* to the triangle ABC *m* is given: Therefore
alſo the reaſon of the double of the rectangle
of FC and AB, to the triangle ABC is given.
But the ſame double of the rectangle of
FC and AB is that, whereof the ſquare of the
line compounded of BAC is greater than the
ſquare of BC, whereof that ſpace of which the
ſquare of the line compounded of BAC is
greater than the ſquare of BC, hath a given
reaſon to the triangle ABC.

OTHERWISE.

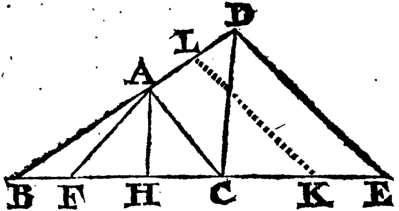
Conſtr. Let the line BA be prolonged to the point D, in ſuch ſort that AD may be equal to AC, and let CD be drawn.

Demonſtr. Foraſmuch as the angle BAC is given, each of the angles ADC and ACD,
which is the half thereof ſhall be alſo given;
and therefore the other angle DAC is alſo
n 40. *prop.* given: Therefore *n* the triangle ACD is given
by kind. Wherefore the reaſon of AC
to CD is given. And foraſmuch as the angle ADC
is given; Let each of the angles DEC and AFC be made † equal to the



the ſaid ADC: Therefore ſeeing that the angle BDC is equal to the angle DEC, and the angle DBE is common to the triangles DBE and DBC, the other angle BDE is equal to the other angle BCD; and therefore the triangle BDE is equiangled to the triangle BDC.
Therefore *o* as EB is to BD, ſo is BD to CB: o 4. 6.
Wherefore the rectangle of EB and CB, that
is to ſay, *p* the rectangle of EC and CB, *q* p 5. 2.
with the ſquare of CB is equal, *r* to the ſquare q 5. 2.
of BD, that is to ſay, to the ſquare of the r 17. 6.
line compounded of BAC; for AD is equal to AC; and therefore the rectangle of EC and CB, with the ſquare of CB, that is to ſay, the ſquare of the line compounded of BAC is greater than the ſquare of the rectangle of BC and CE: I ſay therefore that the reaſon of the ſaid rectangle of BC and CE to the triangle ABC is given. Foraſmuch as the angle BDE is equal to the angle BCD, and the angle ADC equal to the angle ACD, the other angle CDE is equal to the other angle ACB: But the angle DEC is alſo equal to the angle AFC; therefore the remaining angle CAF is equal to the remaining angle DCE. Wherefore the triangle AFC is equiangled to the triangle DCE;
and therefore *s* as CA is to AF, ſo is CD s 4. 6.
to CE; and by permutation, as AC is to CD, ſo is AF to CE. But the reaſon of AC to CD is given: Therefore alſo the reaſon of AF to CE is given. From the point A let AH be drawn perpendicular to BC: Foraſmuch as the angle AFC is given, and the angle AHF alſo given, the third angle HAF
is given; Wherefore *t* the triangle AHF is t 40. *prop.*

given



given by kind; and by conſequence the reaſon of AF to AH is given. But the reaſon of AF to CE is alſo given: u 8. *prop.* Therefore *u* the reaſon of AH to CE is given; and therefore the reaſon of the rectangle x 1. 6. of AH and BC *x* to the rectangle of BC and CE is alſo given. But the reaſon of the rectangle of AH and BC to the triangle ABC is y 41. 1. likewiſe given; (for the rectangle *y* is double to the triangle) and the rectangle of BC and CE is that whereof the ſquare of the line compounded of BAC is greater than the ſquare of BC. Therefore that ſpace of which the ſquare of the line compounded of BAC is greater than the ſquare of BC by a given reaſon to the triangle ADC.

Scholium.

† *The antient Interpreter pretending to ſhew the conſtruction of the angle DEB equal to the angle ADC, ſaith that on the line BD and in the point D, the angle BDE ought to be made equal to the angle BCD, and that the right lines BC and DE be drawn until they interſect in E, in ſuch ſort as he ſuppoſeth the angle BCD, to be given, but it is not.*

The ſame Interpreter afterward ſhews how there may univerſally from a given point, be drawn a right line given by poſition to a right line, making an angle equal to a given angle. But we will alſo reject this way, ſeeing we have elſewhere ſhewn

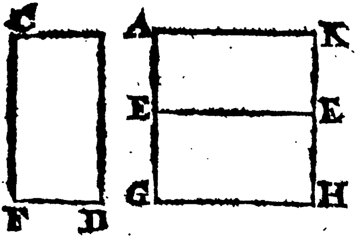
ſhewn another more brief and eaſy. For example, if we would from the point D draw to the line BC given by poſition a right line, making an angle equal to a given angle ADC, as is here required, we have no more to do but to aſſume the point K in the ſaid line BC, and there make the triangle CKL equal to the given angle ADC: If the line KL doth meet with the point D, it ſhall be the line required. But if it meet not with it, from the point D let there be drawn the line DE, parallel to the ſaid KL, cutting BC prolonged in E, and the angle DEC ſhall be equal to the given angle ADC, for on the two parallel lines LK and DE, there doth fall the line BE; and therefore the angle DEC z *is equal to the angle LKC, which* z 29. 1. *hath been made equal to the given angle ADC; and by conſequence the ſame angle DEC is alſo equal to ADC.*

PROP. LXVIII.

If two parallelograms AB and CD, have to one another a given reaſon, and that a ſide hath alſo a given reaſon to a ſide, the other ſide ſhall have likewiſe a given reaſon to the other ſide.

Conſtr. Let the reaſon of BE to FD be given: I ſay the reaſon of AE to FC is alſo given. For to the right line EB let there be applied the parallelogram FH, equal to the parallelogram CD, and conſtituted in ſuch ſort as AE and EG may make one right line: † Therefore KB and BH will alſo make one right line.

Demonſtr.



Demonſtr. Foraſmuch as the reaſon of AB to CD is given, and that EH is equal to the a 1. 6. ſaid CD; the reaſon of AB to EH is given; and therefore the reaſon of AE to EG is alſo given. Seeing therefore that EH is equal and b 14. 6. equiangled to CD, as *b* EB is to FD, ſo is FC to EG. But the reaſon of EB to FD is given: Therefore alſo the reaſon of FC to EG is given. But the reaſon of AE to the ſame EG is alſo given: Therefore the reaſon of AE to FC is given.

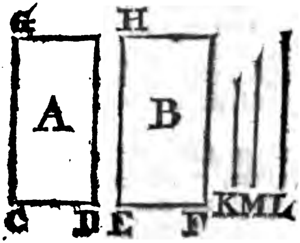
Scholium.

† EUCLIDE having poſited AE and EG directly in one right line, preſently concludeth that KB and BH ſhall alſo make a right line; but we ſhall demonſtrate it thus. Seeing the lines AE and EG are poſited directly, the angles AEB and c 13. 1. *BEG* c *are equal to two right angles; and ſeeing that AB is a parallelogram, the lines AK and EB are parallels, on which the line AE doth fall; and therefore the two internal angles A and BEA* d 29. 1. d *are alſo equal to two right angles, and taking away the common angle BEA, there will remain the angle A, equal to the angle BEG; and conſequently their oppoſite angles EBK and H are alſo equal to one another: Again, ſeeing that BG is a parallelogram, the two lines BE and HG are parallels, on which BH doth fall; and therefore the two internal angles H and EBH* d *are equal to two right angles. But it hath been demonſtrated that*

that H is equal to EBK: Therefore the two angles EBK and EBH are alſo equal to two right angles; and therefore e *the two lines KB and* e 14. 1. *BH do meet directly according to EUCLIDE.*

OTHERWISE.

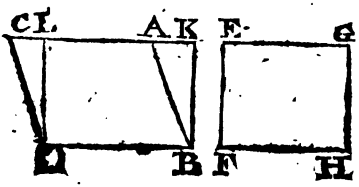
Conſtr. Let the given right line K be expoſed, and ſeeing that the reaſon of A to B is given, let the ſame be made of K to L; therefore the reaſon of K to L is alſo given.



Demonſtr. But K is given; therefore *f* L is f 2. *prop.* alſo given. Again, ſeeing that the reaſon of CD to EF is given, let the ſame be made of K to M: Therefore the reaſon of K to M is given. But K is given, therefore *g* M is alſo g 1. *prop.* given; and therefore the reaſon of L to M is given. Now ſeeing that A is equiangled to B, *h* the reaſon h 23. 6. of the ſaid A to B is compounded of that of the ſides, that is to ſay, of CD to EF, and of CG to EH. But alſo the reaſon of K to L is compounded of K to M, and of M to L; therefore the reaſon compounded of CD to EF, and of CG to EH, is the ſame with that which is compounded of K to M, and of M to L (the reaſon of K to L being the ſame as of A to B:) But the reaſon of CD to EF is the ſame as of K to M: Therefore the other reaſon of CG to EH is alſo the ſame as of M to L. But the ſaid reaſon of M to L is given: Therefore alſo the reaſon of CG to EH is given.

PROP.

PROP. LXIX.



If two parallelograms, CB and EH, having the angles D and F given, and that a ſide hath alſo a given reaſon to a ſide; in like manner the other ſide ſhall have a given reaſon to the other ſide.

Conſtr. Let the reaſon of BD to FH be alſo given: I ſay that the reaſon of AB to EF is given. For if CB be equiangled to HE, it is manifeſt by the precedent Propoſition; but if it be not equiangled thereto, let the right line DB be conſtituted, and in the given point B therein, let the angle DBK be made equal to the angle EFH, and finiſh the parallelogram DK.

Demonſtr. Foraſmuch as each of the angles BKL and BAK is given, † the other angle
a 40.*prop.* KBA is given: Wherefore the triangle *a* ABK is given by kind; and therefore the reaſon of AB to BK is given. But the reaſon of CB to
b 35.*prop.* EH is ſuppoſed to be given, and *b* CB is equal to DK; therefore the reaſon of DK to EH is given; and ſeeing that DK is equiangled to EH, and the reaſon of the ſaid DK to EH is
c 68.*prop.* given, as alſo that of DB to FH, *c* the reaſon of BK to FE is given. But the reaſon of the
d 29. 1. ſaid BK to BA is alſo given: Therefore *d* the reaſon of AB to FE is given.

Scholium.

Scholium.

† *EUCLIDE ſuppoſeth there that a parallelogram having one angle given, all the other angles are alſo given; and as well the antient Interpreters as others, do give the reaſons why, the angle F being given, the other angle E ſhall be alſo given, it being the remainder of two right angles, for that on the parallel lines EG and FH there doth fall the line EF, which makes* e *the two internal angles (of the ſame part) F and G, equal* e 29. 1. *to two right angles. But to thoſe angles* f *the op-* f 34. 1. *poſite angles G and H are equal, and therefore they are alſo given.*

From whence it follows that the angles BDC and F being given by ſuppoſition, all the other angles of the two parallelograms CB and EH, are alſo given: Therefore the angle DBK having been made equal to the angle F, the angle K ſhall be equal to the angle E, and given as that is: But the angle BAL which is oppoſite to the given angle BDC, is alſo given; and therefore BAK which is the remainder of two right angles, ſhall be alſo given; in ſuch ſort as in the triangle ABK, the two angles BAK and BKA are given, as EUCLIDE doth declare in this place.

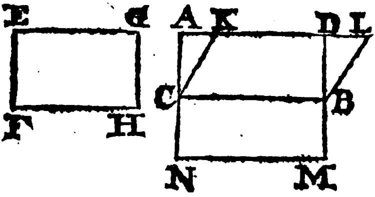
PROP. LXX.

If of two parallelograms AB and EH, the ſides about the equal angles, or about the unequal angles (yet neverthe leſs given angles have to one another a given reaſon, to wit (AC to EF, and CB to FH) alſo the ſame parallelograms AB and EH ſhall have to one another a given reaſon.

Conſtr. For let AB be prolonged to EH, and on the right line CB let the parallelogram CM be applied equal to the parallelogram EH, in

ſuch ſort as AC may be direct to CN; that is to ſay, that AC and CN make one right
a *ſch.* 68. *prop.* line; and by conſequence DB ſhall be *a* directly with BM.

Demonſtr. Foraſmuch then as CM is equiangled and equal to EH, the ſides about the
b 14. 6. equal angles ſhall be reciprocally *b* proportional: Wherefore as BC is to HF, ſo is FE to NC. But the reaſon of BC to HF is given: Therefore the reaſon of FE to NC is alſo given. But the reaſon of AC to the ſame EF is gi-
c 8. *prop.* ven: Therefore *c* the reaſon of AC to NC is alſo given. Wherefore the reaſon of AB to
d 1. 6. CM is given; (for it is the ſame *d* as of AC to CN.) But CM is equal to EH: Therefore the reaſon of AB to EH is given.

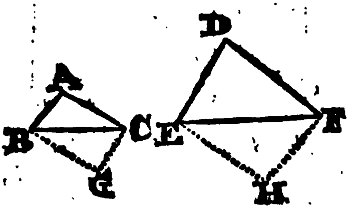


Conſtr. Now ſuppoſe AB not to be equiangled to EH, and on the right line CB, and in the given point C therein: Let there be conſtituted the angle BCK, equal to the given angle F, and ſo finiſh the parallelogram CL.

Demonſtr. Foraſmuch as the angle ACB is given, and the angle BCK alſo given, the remaining angle ACK is given: Therefore the
e 40. *prop.* triangle ACK *e* is given by kind; and therefore the reaſon of AC to CK is given: But the reaſon of AC to EF is alſo given: Therefore the reaſon of CK to EF is given. But the reaſon of BC to HF is alſo given, and the angle BCK is equal to the angle F; therefore

fore (by the firſt part of this Propoſition) the reaſon of CL to EH is given. But to the ſaid CL, AB is equal: Therefore the reaſon of AB to EH is given.

PROP. LXXI.



If of two triangles ABC and DEF, the ſides about the equal angles A and D, or elſe about the unequal angles (yet neverthe-leſs given angles) have to one another a given reaſon (to wit AB to DE and AG to DF) the ſame triangles ſhall have alſo to one another a given reaſon ABC to DEF.

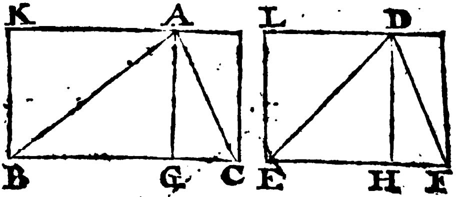
Conſtr. Let the parallelogram AG and DH be finiſhed.

Demonſtr. Seeing that the two parallelograms AG and DH, have the ſides about the equal angles A and D, or elſe about the unequall angles (neverthe-leſs given) have a given reaſon to one another, the reaſon *a* of the parallelogram AG to the parallelogram DH is given. But the triangle ABC is the half of the parallelogram AG *b* and the triangle DEF the half of the parallelogram DH. Therefore the reaſon of the triangle ABC to the triangle DEF is given.

a 70. *prop.*

b 34. *prop.*

PROP. LXXII.



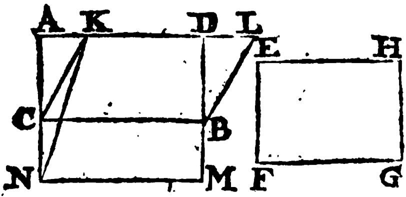
If of two triangles ABC and DEF, the baſes BC and EF, are in a given reaſon, BC to EF, and that from the angles A and D, there be drawn to thoſe baſes the right lines AG and DH, making the equal angles AGC and DHF, or elſe unequal (yet nevertheleſs given) which ſhall have to one another given reaſons AG to DH, thoſe triangles ABC and DEF ſhall have alſo a given reaſon to one another, to wit, ABC to DEF.

Conſtr. For let the parallelogram KC and LF be finiſhed.

Demonſtr. Foraſmuch as the angles AGC and DHF are equal, or unequal (yet given) and
a 29. 1. that the angle AGC *a* is equal to the angle KBC. But the angle DHF equal to the angle LEF, the angles at the points B and E are equal, or elſe unequal (yet given) and for that the reaſon of AG to DH is given, and AG is equal to KB. But DH is equal to LE, alſo the reaſon of KB to LE is given. But the reaſon of BC to EF is alſo given, and the angles at the points B and E are equal, or
b 70. prop. elſe unequal (yet given:) Therefore *b* the reaſon of the parallelogram KC to the parallelogram LF is given; and therefore the reaſon of the triangle ABC to the triangle DEF is gi-
c 41. 1. ven, ſeeing thoſe triangles *c* are the one half of the parallelograms.

PROP.

PROP. LXXIII.



If of two parallelograms AB and EG, the ſides about the equal angles C and F, or elſe about the unequal angles (but neverthe leſs given) are in ſuch ſort to one another, that as the ſide CB of the firſt, is to the ſide FG of the ſecond; ſo the other ſide EF of the ſecond, is to ſome other right line CN. But that the other ſide AC hath alſo to the ſame right line CN a given reaſon, thoſe parallelograms will have alſo to one another a given reaſon AB to EG.

Conſtr. For in the firſt place, let the parallelogram AB be equiangled to EG, and having placed CN directly to AC: Let the parallelogram CM be finiſhed.

Demonſtr. Foraſmuch then as CB or NM its equal, is to FG, ſo is EF to CN, and that the angles N and F are equal (for N is equal to the angle ACB, which is put equal to F) the parallelograms CM and EG *a* are equal: a 14. 6.
But as AC to CN, ſo *b* the parallelogram AB b 1. 6.
is to the parallelogram CM or EG: Therefore ſeeing that the reaſon of AC to CN is given, the reaſon of AB to EG is alſo given.

Conſtr. 2. Now ſuppoſe the parallelogram AB not to be equiangled to the parallelogram EG, and let there be conſtituted at the given point C in the line CB, the angle BCK, equal to the angle EFG, and ſo finiſh the parallelogram CL.

Demonſtr. 2. Seeing that each of the angles ACB and KCB is given, the remaining angle c *ſch.* 69. *prop.* ACK is alſo given. But *c* the angle CAK is given, as alſo the remaining angle AKC: d 40. *prop.* Therefore *d* the triangle ACK is given by kind; and therefore the reaſon of AC to CK is given. But the reaſon of the ſame AC to e 8. *prop.* CN is alſo given: Therefore *e* the reaſon of CK to CN is given. And ~~ſeeing that~~ as CB is to FG; ſo is EF to the right line CN; to which the other ſide KC hath a given reaſon, and that the angle BCK is equal to the angle F, the reaſon of the parallelogram CL to the parallelogram EG is given (by the firſt part of this Propoſition) but the parallelogram CL is equal to the parallelogram AB: Therefore the reaſon of the parallelogram AB to the parallelogram EG is given.

PROP. LXXIV.

If two parallelograms (as in the former figure) AB and EG, in equal angles C and F, or elſe in unequal angles (yet nevertheleſs given angles) have a given reaſon to one another, as one ſide CB of the firſt ſhall be to one ſide FG of the ſecond, ſo the other ſide EF of the ſecond, ſhall be to that to the which the other ſide AC of the firſt hath a given reaſon.

Conſtr. For either AB is equiangled or not; ſuppoſe it in the firſt place to be equiangled, and to the right line BC let there be applied the parallelogram CM, equal to the parallelogram EG, and ſo poſited, as that a *ſch.* 68. *prop.* AC and CN may be direct: Therefore *a* DB and BM ſhall be alſo direct (that is as one right line.)

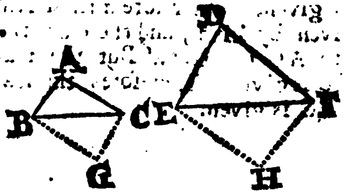
Demonſtr. Seeing that the reaſon of AB to EG is given, and that CM is equal to EG, the reaſon of AB to CM is alſo given; and therefore the reaſon of AC to CN is given

(ſeeing AB is to CM, *b* as AC is to CN;) and b 1. 6.
for that CM is equal and equiangled to EG,
the ſides about the equal angles of the parallelograms CM and EG, *c* are reciprocally pro- c 14. 6.
portional; and therefore as CB is to FG, ſo
is EF to CN. But the reaſon of AC to CN
is given: Therefore as CB is to FG, ſo is
EF to that to which AC hath a given reaſon.

Conſtr. 2. Now ſuppoſe AB not to be equiangled to EG, and in the given point C of the line CB, let there be conſtituted the angle BCK equal to the angle EFG, and finiſh the parallelogram CL.

Demonſtr. 2. Seeing then that the reaſon of
AB to EG is given, and *d* that AB is equal d 36. 1.
to CL, alſo the reaſon of CL to EG is given,
and the angle BCK is equal to the angle F,
and therefore CL *e* is equiangled to EG: e *ſch.* 69.
Therefore (by the firſt part of this Propoſi- *prop.*
tion) as CB is to FG, ſo is EF to that to
the which CK hath a given reaſon. But the
reaſon of AC to CK is given; (as appears
by what hath been demonſtrated in the latter
part of the precedent Propoſition.) Therefore
as CB is to FG, ſo is EF to that to which
AC hath a given reaſon.

PROP. LXXV.



If two triangles ABC and DEF, in equal angles A and D, or elſe unequal (yet nevertheleſs given) have to one another a given reaſon, as the ſide AB of the firſt, ſhall be to the ſide DE of the ſecond, ſo the other

other ſide DF of the ſecond, ſhall be to that right line to the which the other ſide AC of the firſt hath a given reaſon.

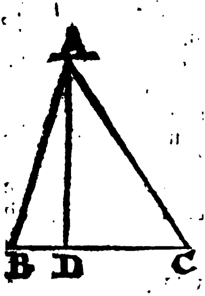
Conſtr. For let the parallelograms AG and DH be finiſhed.

Demonſtr. Foraſmuch as the reaſon of the triangle ABC to the triangle DEF is given, alſo the reaſon of the parallelogram AG to the parallelogram DH is given.

Seeing therefore that the two parallelograms AG and DH in equal angles, or unequal angles (nevertheleſs given) have to one another a given reaſon; as *a* AB is to DE, ſo is DF to that to which AC hath a given reaſon.

a 74 *prop.*

PROP. LXXVI.



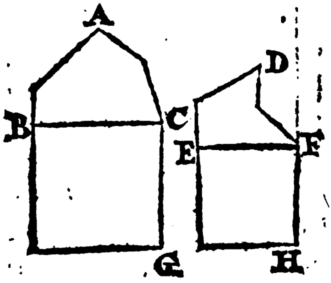
If from the top A of a triangle ABC, given by kind, there be drawn to the baſe BC, a perpendicular line AD, that line AD ſhall have to the baſe BC a given reaſon.

Demonſtr. For ſeeing that the triangle ABC is given by kind, the reaſon of AB to BC is given; and the angle B is alſo given. But the angle ADB is given; therefore the other angle BAD is given. Wherefore *a* the triangle ABD is given by kind; and therefore the reaſon of AB to AD is given. But the reaſon of AB to BC is given: Therefore *b* the reaſon of AD to BC is given.

a 40. *prop.*
b 8. *prop.*

PROP. LXXVII.

If two figures ABC and DEF, given by kind, have to another a given reaſon, the reaſon alſo ſhall be given of which you pleaſe of the ſides of one of the figures, to which you pleaſe of the ſides of the other figure.



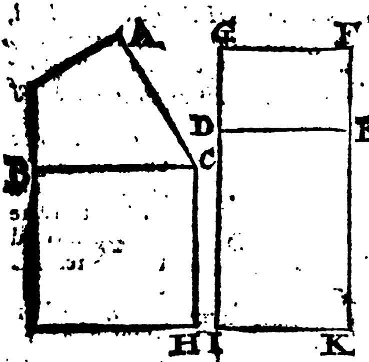
Conſtr. For on the right lines BC and EF, let there be deſcribed the ſquares BG and EH.

Demonſtr. For-aſmuch as on one and the ſame right line BC, are deſcribed two figures ABC and BG, given by kind, *a* the reaſon of the ſaid ABC to BG is given. In like manner, the reaſon of DEF to EH is given; and ſeeing that the reaſon of ABC to DEF is given, and alſo that of the ſame figure ABC to BG; and again the reaſon of DEF to EH: *b* the reaſon of BG to EH is given; and therefore the reaſon of BC to EF is alſo given.

a 49. *prop.*

b 8. *prop.*

PROP. LXXVIII.



If a given figure ABC, hath a given reaſon to ſome rectangled figure DF, and that one ſide BC hath a given reaſon to one ſide DE, the rectangled figure DF is given by kind.

Conſtr. For on the right line BC let the ſquare BH be deſcribed, and to the right line DE, let the parallelogram DK be applied equal to BH, in ſuch manner, as that GD and DI may be placed directly, *a* and by conſequence FE and EK alſo directly. a *ſch.* 68. *prop.*

Demonſtr. Therefore ſeeing that on one and the ſame right line BC are deſcribed the two rectiline figures ABC and BH, given by kind, *b* the reaſon of ABC to BH is given. But the reaſon of the ſaid ABC to DF is alſo given: Therefore *c* the reaſon of BH to DF is given. But BH is equal to DK: Therefore the reaſon of DK to DF is alſo given. And ſeeing that BH is equal and equiangled to DK, both the one and the other being rectangles, *d* the ſides of thoſe figures are reciprocally proportional; and as BC is to DE, ſo is DI to CH. But by ſuppoſition, the reaſon of BC to DE is given; therefore alſo the reaſon of DI to CH is given; but the reaſon of

b 49. *prop.*
c 8. *prop.*
d 14. 6.

of DI to DG is alſo given; (for DI is to DG e as DE to DF;) Therefore f the reaſon of DG to CH is given. But CH is equal to BC, ſeeing that BH is a ſquare; therefore the reaſon of BC to DG is given. But the reaſon of the ſame BC to DE is alſo given; therefore the reaſon of DE to DG is given, and the angle at D is a right angle: Therefore g DF is given by kind.

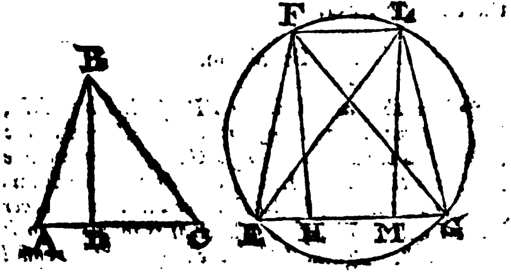
e 1. 6.

f 8. prop.

g ſch. 61. prop.

PROP. LXXII.

If two triangles ABC and EFG, have an angle B equal to an angle F. But from the equal angles B and F there be drawn perpendiculars BD and FH, to the baſes AC and EG; and that as the baſe AC of the firſt triangle ABC, is to the perpendicular BD, ſo alſo the baſe EG of the other triangle EFG, is to the perpendicular FH, thoſe triangles ABC and EFG are equiangled.



Conſtr. For about the triangle EFG let there be deſcribed the circle EFLG, then on the right line EG, and in the point E given therein, let there be made the angle GEL, equal to the angle C, and let FL and LG be drawn, and the perpendicular LM.

Demonſtr. Seeing then that the angle GEL
is equal to the angle C, and the angle ELG
a 21. 3. is equal to the angle EFG, *a* they being in
one and the ſame ſegment of the circle; the
third angle EGL is equal to the third angle
A. Wherefore the triangle ABC is alike to
the triangle ELG, and the perpendiculars BD,
and LM are drawn: Therefore † as AC is
to BD, ſo is EG to LM; but by ſuppoſition
as AC is to BD; ſo is EG to FH: Therefore
b 7. 5. *b* LM is equal to FH. But the ſaid LM
c 28. 1. is *c* parallel to FH: Therefore *d* FL is alſo pa-
d 33. 1. rallel to EG; and therefore the angle FLE
e 29. 1. *e* is equal to the angle LEG. But the an-
gle C is alſo equal to the ſaid angle LEG, and
f 21. 3. the angle FLE to the angle FGE *f*: Therefore
alſo the angle C is equal to the angle FGE.
But by ſuppoſition the angle ABC is equal
to the angle EFG: Therefore the third an-
gle BAC is equal to the third angle FEG:
Wherefore the triangle ABC is equiangled to
the triangle EFG.

Scholium.

† *Now that as AC is to BD, ſo EG is to LM,*
it is by ſome thus demonſtrated. Foraſmuch as the
angle C is equal to the angle GEL, and the angle
BDC to the angle LME, each being a right an-
gle, the other angle CBD is equal to the other
g 4. 6. *angle ELM: Therefore* g *as EM is to ML, ſo is*
CD to DB. Again, ſeeing the angle ABC is equal
to the angle ELG, and the angle CBD to the
angle ELM, the remaining angle ABD is equal
to the remaining angle MLG; but the angle
ADB is alſo equal to the angle LMG; and there-
fore the third angle A is equal to the third angle
h 4. 6. *LGM: Therefore* h *as AD is to DB, ſo is GM*
to ML. But it hath been demonſtrated that as

CD is to DB, ſo is EM to ML: Therefore i *as* i 14. 5.
AC is to BD, ſo is EG to LM.

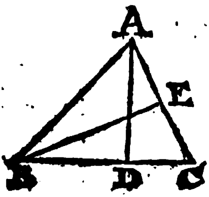
PROP. LXXX.

If a triangle ABC hath one angle A given, and that the rectangle contained under the ſides AB and AC, compriſing the given angle A, hath a given reaſon to the ſquare of the other ſide BC, the triangle ABC is given by kind.

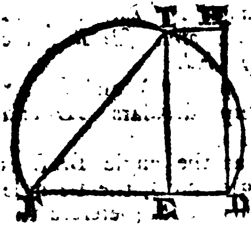
Conſtr. For from the points A and B, let there be drawn the perpendiculars AD and BE.

Demonſtr. Foraſmuch as the angle BAE is given, and alſo the angle AEB, the triangle ABE is given by *a* kind; and therefore the reaſon of AB to BE is given: Therefore the reaſon of the rectangle of AB and AC to the rectangle of BE and AC is alſo given (for it is the ſame reaſon *b* as of AB to BE.) But the rectangle of AC and BE is equal to the rectangle of BC and AD; for that each of thoſe rectangles is *c* double to the triangle ABC. Therefore the reaſon of the rectangle of AB and AC to the rectangle of BC and AD is alſo given. But the reaſon of the rectangle of AB and AC to the ſquare of BC is given: Therefore *d* alſo the reaſon of the rectangle of BC and AD to the ſquare of BC is given; and therefore the reaſon of the right line BC to the right line AD is given. (For that *e* the rectangle is to the ſquare as AD to BC.) Now let the right line FG given by poſition and magnitude, be expoſed; and thereon let there be deſcribed the ſegment of a circle FIG; capable

a 40. *prop.*
b 1. 6.
c 41. 1.
d 8. *prop.*
e 1. 6.



capable of an angle equal to the angle A.
And ſeeing the ſaid angle A is given, alſo the
angle in the ſegment FLG ſhall be given ;
f 8. *def.* and therefore *f* the ſame ſegment is given by
poſition. From the point G let there be erect-
ed at right angles on the line FG, the line
g 4. *def.* GH, which *g* is given by poſition : Let
it be ſo made, that as BC is to AD, ſo FG
may be to GH ; and ſeeing that the reaſon
of BC to AD is given, alſo that of FG to
GH is given. But FG is given : There-
h 2. *prop.* fore *h* GH is given by
magnitude. But it is alſo given by poſition,
and the point G is given : Therefore the point
i 27. *prop.* H is *i* alſo given. Now by the point H let
there be drawn HI, parallel to FG, and that
k 28. *prop.* line HI ſhall be given by *k* poſition. But the
ſegment of the circle FIG is alſo given by
l 25. *prop.* poſition. Therefore *l* the point I is given.
Let the right lines IF and IG be drawn, and
the perpendicular IK ; Therefore IK is given
by poſition. But the point I is given, as alſo
m 26. *prop.* each of the points F and G : Therefore *m*
each of the lines FG, FI, and IG is given
n 39. *prop.* by poſition and magnitude : Wherefore *n* the
triangle FIG is given by kind ; and ſeeing
that as BC is to AE, ſo is FG to GH, and
o 34. *prop.* *o* that to GH, IK is equal, as BC is to AE,
ſo is FG to IK, and the angle A is equal to
p 79. *prop.* the angle FIG : Therefore *p* the triangle ABC
is equiangled to the triangle FIG. But FIG
is given by kind : Therefore alſo the triangle
ABC is given by kind.



OTHER

OTHERWISE.

Conſtr. Let the triangle ABC, whoſe angle A is given, and the reaſon of the rectangle contained under AB and AC, to the ſquare of BC be given: I ſay that the triangle ABC is given by kind.

Demonſtr. For ſeeing the angle A is given, that ſpace of which the ſquare of the line compounded of BAC is greater than the ſquare of BC, *q* hath a given reaſon to the triangle ABC. Now let that ſpace be D: Therefore the reaſon of D to the triangle ABC is given. But the reaſon of the triangle ABC to the rectangle of AB and AC is given; *r* ſeeing the angle A is given: Therefore *s* the reaſon of the ſpace D to the rectangle of AB and AC is given. But the reaſon of the rectangle of AB and AC to the ſquare of BC is alſo given: Therefore *s* the reaſon of the ſpace D to the ſquare of BC is given. Wherefore by compounding, *t* the reaſon of the ſpace D, with the ſquare of BC to the ſaid ſquare of BC is given: Therefore the reaſon of the ſquare of the line compounded of BAC, to the ſquare of BC is given; (for that the ſpace D with the ſquare of BC is equal to the ſquare of the line compounded of BAC;) and therefore *u* the reaſon of the ſaid line compounded of BAC to BC is given. But the angle A is alſo given: There-

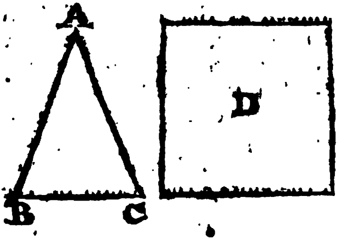
q 67. *prop.*

r 66. *prop.*

s 8. *prop.*

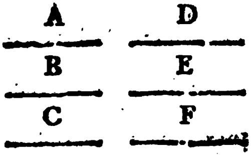
t 6. *prop.*

u *ſch.* 52. *prop.*



x 46. *prop.* Therefore *x* the triangle ABC is given by kind.

PROP. LXXXI.



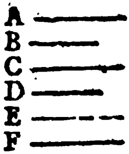
If of three right lines A, B, and C, proportional to three other proportional right lines D, E, and F, the extremes A and D, C and F, are in a given reaſon (to wit, as A to D, and C to F,) alſo the means B and E ſhall be in a given reaſon, and if one extreme hath a given reaſon to an extreme, and the mean to the mean, the other will have alſo a given reaſon to the other.

Demonſtr. Foraſmuch as the reaſon of A to D, and of C to F is given, the rectangle of A a 70. *prop.* and D *a* ſhall have a given reaſon to the rectangle of C and F. But the rectangle of A and b 17. 6. D is equal *b* to the ſquare of B; and the rectangle of C and F to the ſquare of E. Therefore the reaſon of the ſquare of B to the ſquare of c *ſch.* 52. *prop.* E is given; and therefore *c* the reaſon of the line B to the line E is alſo given.

Again, Let the reaſon of A to D, and B to E, be given: I ſay that the reaſon of C to F is alſo given. For ſeeing that the reaſon of A to D, and of B to E is given, alſo the reaſon d 50. *prop.* of the ſquare of B *d* to the ſquare of E is given. But the ſquare of B is equal to the rectangle of A and C, and the ſquare of E to the rectangle of D and F: Therefore the reaſon of the rectangle of A and C to the rectangle of D and F is given. But the reaſon of a ſide A to a ſide D is given: Therefore *e* the reaſon of e 68. *prop.* the other ſide C to the other ſide F is alſo given.

PROP.

PROP. LXXXII.

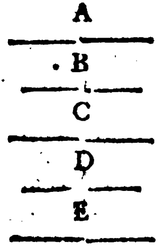


If there be four right lines A,B,C, and D, proportional, as the firſt A, ſhall be to that line to which the ſecond B hath a given reaſon, ſo the third C, ſhall be to that to which the fourth D hath a given reaſon.

Conſtr. Let E be the line to which B hath a given reaſon, and let it be ſo as that B may be to E, as D is to F.

Demonſtr. Now the reaſon of B to E is given, therefore alſo the reaſon of D to F is given. And ſeeing that as A is to B, ſo is C to D. And again, as B is to E, ſo is D to F, by reaſon of equality as A is to E, ſo is C to F. But E is that line to which B hath a given reaſon, and F that to which D alſo hath a given reaſon: Therefore as A is to that to which B hath a given reaſon, ſo C is to that to which D hath a given reaſon.

PROP. LXXXIII.



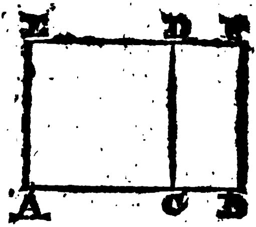
If four right lines A, B, C, and D, are in ſuch ſort to one another, that of any three of them A, B, and C, and a fourth E, taken proportional, to which that line D, which remains of the four lines, hath a given reaſon, the four lines A, B, C, and E, are proportional; as the fourth D is to the third C, ſo the ſecond B ſhall be to that to which the firſt A hath a given reaſon.

Demonſtr. Foraſmuch as A is to B as C is to E, the rectangle contained under A and E *a* is equal to the rectangle contained under B and [a] 16.6

C;

C; and ſeeing that the reaſon of D to E is given, alſo ſhall be given the reaſon of the rectangle of A and D to the rectangle of A and b 1.6. E (for *b* it is the ſame reaſon as of D to E.) But the rectangle of A and E is equal to the rectangle of B and C: Therefore the reaſon of the rectangle of A and D to the rectangle of B c 74.*prop.* and C is given. Wherefore *c* as D is to C, ſo is B to that to which A hath a given reaſon.

PROP. LXXXIV.

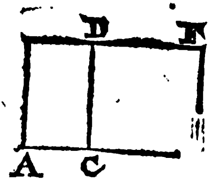


If two right lines AB and AE comprehending a given ſpace AF in a given angle BAE, and that the one AB be greater than the other AE by a given line CB, alſo each of the lines AB and AE is given.

Demonſtr. For ſeeing that AB is greater than AE by the given line CB, the remainder AC is equal to AE: Finiſh the parallelogram AD. Therefore ſeeing that AE is equal to AC, the reaſon of AE to AC is given, and the angle a *ſch.* 61. *prop.* 5 A is alſo given: Therefore *a* AD is given by kind. Wherefore the given ſpace AF is applied to the given right line CB, exceeding it by the given figure AD given by kind; and therefore b 59.*prop.* *b* the breadth of the exceſs is given. Therefore AG is given. But CB is alſo given: Therefore the whole AB is given. But AE is alſo given: Therefore each of the right lines AB and AE is given.

PROP.

PROP. LXXXV.



If two right lines AC and CD, do comprehend a given ſpace AD in a given angle ACD, the line compounded of thoſe lines AC and CD is given, alſo each of thoſe lines AC and CD is given.

Conſtr. For let AC be prolonged to the point B, and let CB be put equal to CD, then by the point B let BF be drawn parallel to CD, and ſo finiſh the parallelogram CF.

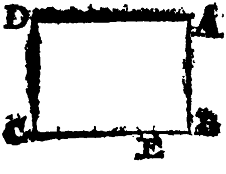
Demonſtr. Seeing then that CB is equal to CD, and the angle DCB is given; for that angle that follows is the given angle; and therefore *a* the parallelogram DB is given by kind; and again, ſeeing that the line compounded of ACD is given, and CB is equal to CD, alſo AB is given. And thus to the right line AB there is applied the given ſpace AD, deficient by the figure DB given by kind; and therefore *b* the breadths of the defects are alſo given: Therefore the right lines DC and CB are given. But the compounded line ACD is alſo given: Therefore *c* each of the lines AC and CD is given.

a *ſch.* 61. *prop.*

b 58. *prop.*

c 4. *prop.*

PROP. LXXXVI.



If two right lines AB and BC, do comprehend a given ſpace AC, in a given angle ABC, the ſquare of the one BC, is greater than the ſquare of the other AB, by a given ſpace (yet in a given reaſon) alſo each of theſe lines AB and BC ſhall be given.

Demonſtr. For ſeeing that the ſquare of BC
is greater than the ſquare of AB by a given
ſpace (yet in a certain reaſon.) Let the given
ſpace be taken away, that is to ſay, the rect-
angle contained under CB and BE: Therefore
a 11. *def.* *a* the reaſon of the remainder, *b* which is the
b 2. 2. rectangle contained under BC and CE to the
ſquare of AB is given. And foraſmuch as the
rectangle † under AB and BC is given, and
c 1. *prop.* alſo that of CB and BE, their *c* reaſon is given.
But as the rectangle under AB and BC is to
d 1. 6. the rectangle under CB and EB, *d* ſo AB is to
BE; and therefore the reaſon of AB to BE
e 50. *prop.* is given: Wherefore *e* the reaſon of the ſquare
of AB to the ſquare of BE is alſo given. But
the reaſon of the ſquare of AB to the rectan-
f 8. *prop.* gle under BC and CE is given: Therefore *f*
alſo the reaſon of the rectangle under BC and
CE to the ſquare of BE is given. Wherefore
the reaſon of four times the rectangle under
BC and CE to the ſquare of BE is given;
g 8. *prop.* and by compounding, *g* the reaſon of four
times the rectangle under BC and CE, with
the ſquare of BE to the ſquare of BE is given.
But four times the rectangle of BC and CE,
h 8. 2. with the ſquare of BE, *h* is the ſquare of the
compound line BCE: Therefore the reaſon of
the

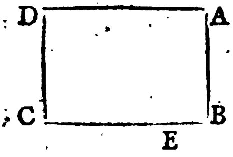
ſquare of the compound line BCE to the ſquare
of BE is given: Wherefore i the reaſon of the i 54. *prop.*
line compounded of BC and CE to BE is gi-
ven, and by compounding, k the reaſon of the k 6. *prop.*
compound of the lines BC, CE, and BE, that
is to ſay, the double of BC to BE is given;
and therefore the reaſon of the only line BC
to BE is alſo given. But as BC is to BE, l ſo l 1. 6.
the rectangle under BC and BE is to the ſquare
of BE: Therefore the reaſon of the rectangle
under BC and BE to the ſquare of BE is gi-
ven But the rectangle of BC and BE is gi-
ven: Therefore m the ſquare of BE is alſo gi- m 2. *prop.*
ven, and conſequently the line BE is given.
Wherefore BC is alſo given, ſeeing that the
reaſon of BE to BC is given. But the ſpace
AC is given, and alſo the angle B †. Therefore
n AB is given. Wherefore each of the lines n 57. *prop.*
AB and BC is given.

Scholium.

† *Inſtead of ſaying in this place* [what is under, &c.] *we have uſed this Word Rectangle, it being manifeſt by what follows that ſuch was the intention of EUCLIDE, ſeeing he makes uſe in the ſaid Demonſtration of the ſecond and eighth Propoſition of the twelfth Element; and alſo that the ſpace or Parallelogram given being not rectangled, it may be reduced thereto, making on BC, and in the given point B, a right angle CBA, ſo as that there will be two Parallelograms conſtituted on one and the ſame baſe BC, and between the ſame parallels, as in the 69th Propoſition by means whereof this Concluſion is drawn.*

Note, This ſerves alſo for the next Prop.

PROP. LXXXVII.



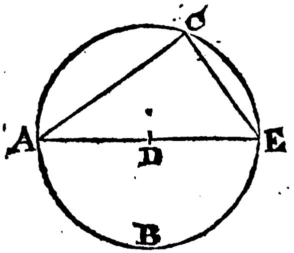
If two right lines AB and BC, do comprehend a given ſpace AC, in a given angle B, the ſquare of the one BC, is greater than the ſquare of the other AB, by a given ſpace, alſo each of thoſe lines AB and BC ſhall be given.

Demonſtr. For ſeeing that the ſquare of BC
is greater than the ſquare of AB by a given
ſpace: Let the given ſpace be taken away, and
let the rectangle be contained under BC and
a 2. 2. BE: Therefore the remainder, *a* which is the
rectangle of BC and CE, is equal to the ſquare
of AB. And ſeeing that the rectangle of BC
and BE is given, and alſo the ſpace or rectangle AC; the reaſon of the ſaid rectangle of
b 1. 6. BC and BE to AC is given. But as *b* the rectangle of BC and BE is to the rectangle of
AB and BC, ſo is BE to AB: Therefore the
reaſon of BE to AB is given, and therefore
c 50. *prop.* *c* the reaſon of the ſquare of the ſaid DE to
the ſquare of AB is alſo given. But to that
ſquare of AB the rectangle of BC and CE is
equal: Therefore the reaſon of the ſaid rectangle of BC and CE to the ſquare of BE is given; and therefore the reaſon of the quadruple
of the ſaid rectangle of BC and CE to the
ſquare of BE is alſo given; and by compound-
d 6. *prop.* ing, *d* the reaſon of four times the rectangle
of BC and CE, with the ſquare of BE, to the
ſaid ſquare of BE is given. But four times
the rectangle of BC and CE, with the ſquare
e 8. 2. of BE, *e* is the ſquare of the compound line
BCE: Therefore the reaſon of the ſquare of
that compound line BCE to the ſquare of BE

is alſo given; and therefore the reaſon *f* of the compound line BCE to BE is given. Wherefore by compounding, *g* the reaſon of the ſaid compound line BCE and EB, that is to ſay, twice BC to BE is alſo given; therefore the reaſon of the only line BC to BE is given. But the reaſon of the ſame BE to AB is alſo given: Therefore *h* the reaſon of AB to BC is given. And ſeeing that the reaſon of BC to BE is given, and that as the ſaid BC is to BE, ſo the ſquare of BC *i* to the rectangle of BC and BE, the reaſon of the ſquare of BC to the rectangle of BC and BE is alſo given. But the ſaid rectangle of BC and BE is given, it being that which was taken away, and which was given. Therefore the ſquare of BC *k* is given, and therefore the line BC is given. But the reaſon of the ſame BC to BA is given, therefore AB is alſo given.

f 54. *prop.*
g 6. *prop.*
h 8. *prop.*
i 1. 6.
k 2. *prop.*

PROP. LXXXVIII.



If in a circle ABC, given by magnitude, there be drawn a right line AC, which ſhall take away a ſegment ABC, which doth comprehend a given angle AEC, that line AC is given by magnitude.

Conſtr. For let D be the center of the circle; and let the diameter thereof ADE be drawn, and let EC be joined.

Demonſtr. Foraſmuch as the angle ACE is given, for *a* it is a right angle. But the angle AEC is alſo given, and therefore the other angle CAE is given. Wherefore the triangle

a 31. 3.

b 40. *prop.* ACE *b* is given by kind; and therefore the
reaſon of EA to AC is given. But AE is gi-
ven by magnitude, ſeeing that the circle ABC
c 2. *prop.* is given by magnitude. Therefore *c* AC is
alſo given by magnitude.

PROP. LXXXIX.

If in a circle ABC, given by magnitude, there be drawn a right line AC, given by magnitude, that line AC will take away a ſegment ABC, comprehending a given angle.

Conſtr. For having taken the point D for the center of the circle, let the diameter ADE be drawn, as alſo the right line EC.

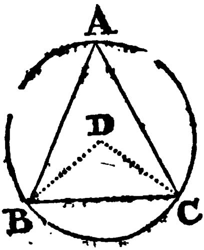
Demonſtr. Foraſmuch as each of the right
lines AE and AC are given, the reaſon of the
a 1. *prop.* line AE to AC *a* is given; and the angle ACE
b 43. *prop.* is a right angle: Therefore *b* the triangle ACE
is given by kind, and therefore the angle AEC
is given.

PROP. XC.

If in the circumference of a circle ABC given by poſition, and by magnitude, there be taken a given point B, and that from that point B, to the circumference of the circle ABC, there doth bend a right line BAC, making a given angle BAC the other extremity C of the bent line ſhall be given.

Conſtr. For let the center of the circle be D, and let the right lines BD and BC be drawn.

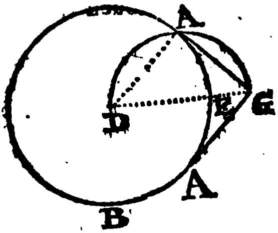
Demonſtr.



Demonſtr. Foraſmuch as each point B and D is given, the right line BD, *a* is given by poſition; and ſeeing that the angle BAC is given, the angle BDC is alſo given. Wherefore to the right line BD given by poſition, and in the point D given therein, there is drawn the right line CD; which makes the given angle BDC; and therefore *b* the line DC is given by poſition. But the circle ABC is given by poſition and magnitude: Therefore *c* the right line DC is given by poſition and by magnitude. But the point D is given: Therefore *d* the point C is alſo given.

a 26. *prop.*
b 29. *prop.*
c 6. *def.*
d 27. *prop.*

PROP. XCI.



If from a given point C, there be drawn a right line CA, which ſhall touch a circle AB, given by poſition; that line CA is given by poſition and by magnitude.

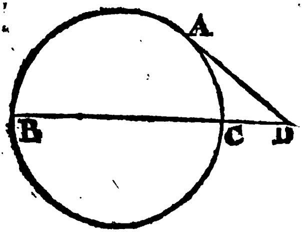
Conſtr. For having taken the point D for the center of the circle, let the right lines DA and DC be drawn.

Demonſtr. Foraſmuch as each point C and D is given, the right line CD *a* is given by poſition and by magnitude. But the angle CAD *b* is a right angle; and therefore the ſemicircle deſcribed on CD ſhall paſs by the point A: Let it then paſs by that point, and let the ſame ſemicircle be DAC: Foraſmuch as the ſame

a 26. *prop.*
b 18. 3.

c 6. *def.* DAC *c* is given by poſition, and alſo the circle
d 25. *prop.* ABE, *d* the point A is given. But the point
e 26. *prop.* C is alſo given: Therefore *e* the right line
AC is given by poſition and by magnitude.

PROP. XCII.

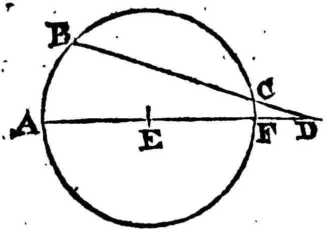


If without a circle ABC, given by poſition, there be taken ſome point D, and from that given point there be drawn a right line DB, cutting the circle, the rectangle compriſed under the whole line BD, and the part DC, between the point D, and the circumference convex AC ſhall be given.

Conſtr. For from the point D let the right line DA be drawn, which ſhall touch the circle in the point A.

a 91. *prop.* *Demonſtr.* Therefore DA *a* is given by poſition and magnitude; and therefore the ſquare
b 52. *prop.* of the ſaid DA is *b* given. But the ſaid
c 36. 3. ſquare of DA is equal *c* to the rectangle of BD and DC: Therefore the ſaid rectangle of BD and DC is alſo given.

OTHERWISE.



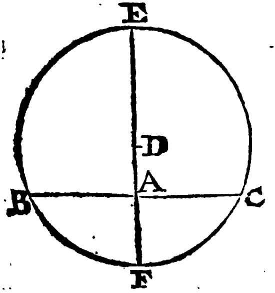
Conſtr. Let E be the center of the circle, and by the ſame center let there be drawn from the point D the right line DA.

Demonſtr. Foraſmuch as each point D and E is given, the right line DE is *d* given by poſition and by magnitude. But the circle ABC is given by poſition and by magnitude: Therefore each point A and F *e* is given, and the point D is alſo given; and therefore *d* each line AD and FD is given. Wherefore the rectangle of the lines AD and DF is alſo given. But the ſaid rectangle of AD and DF is equal to the rectangle of DB and DC: Therefore the rectangle of DB and DC is given.

d 26.*prop.*

e 25.*prop.*

PROP. XCIII.



If in a circle given by poſition there be taken a given point A, and by that point A there be drawn a right line BC to the circle, the rectangle compriſed under the ſegments of the ſame line BC ſhall be given.

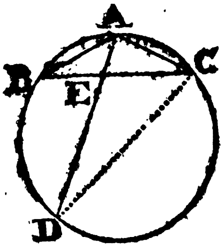
Conſtr. For let D be taken for the center of the circle, and having drawn the right line AD prolong it to the points E and F.

Demonſtr. Foraſmuch as each point A and D is given, the right line AD *a* is given by poſition. But the circle BEC is alſo given by poſition: Therefore each point E and F is alſo given by poſition, and the point A is given. Wherefore each line *b* AE and AF is given: Therefore the rectangle of the ſame lines AE and AF is given; and is equal to the rectangle *b* of AB and AC: Therefore the ſaid rectangle of AB and AC is given.

a 26. *prop.*

b 35. 3.

PROP. XCIV.



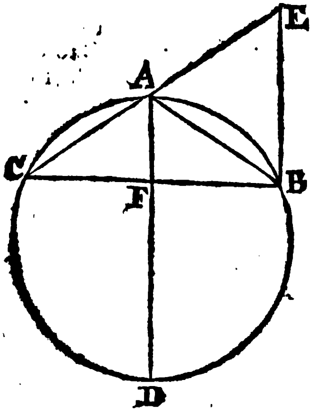
If in a circle ABC, given by magnitude, there be drawn a right line BC, which doth take away a ſegment which doth comprehend a given angle ABC, and that the ſaid angle being in the ſegment is cut into two equal parts, the line compounded of the right lines BA and AC, which comprehend the given angle BAC ſhall have a given reaſon to the line AD, which doth divide that angle into two equal parts; and the rectangle contained under the line compounded of thoſe lines BA and AC, comprehending the given angle BAC, and that part ED of the interſecting line which is below the ſegment between the baſe BC and the circumference, ſhall be given.

Conſtr. Let BD be drawn.

Demonſtr. Foraſmuch as in the circle ABC given by magnitude, there is drawn the right line BC, which takes away the ſegment BAC comprehending the given angle BAC, that

line BC a is given; and therefore BD is alſo a 88. prop.
given: Therefore the reaſon of BC to BD b is b 1. prop.
given. And ſeeing that the given angle BAC
is cut in two equal parts by the right line AD, as
c BA is to CA, ſo is BE to CE; and by compounding, c 3. 6.
as BAC is to CA, ſo is BC to CE;
and by permutation, as BAC is to BC, ſo is
CA to CE. And ſeeing that the angle BAE
is equal to the angle CAE, and the angle
ACE d to the angle BDE, the other angle d 21. 3.
AEC is equal to the other angle ABD; and
therefore the triangle ACE is equiangled
to the triangle ABD: Therefore e as AC is to e 4. 6.
CE, ſo is AD to BD. But as AC is to CE,
ſo the line compounded of BA and AC is to
BC: Therefore as the compound line BAC is
to BC, ſo is AD to BD; and by permutation,
as the compound line BAC is to AD, ſo is
BC to BD. But the reaſon of BC to BD is
given: Therefore the reaſon of the compound
line BAC to AD is alſo given. Moreover,
I ſay that the rectangle under the compound
line BAC and ED is given. For ſeeing that
the triangle AEC is equiangled to the triangle
BDE, (for the angle ACE d is equal to the
angle BDE, and the angle AEC f to the angle f 15. 1.
BED) as BD is to DE, ſo is AC to CE. But
as AC is to CE, ſo is alſo the compound
line BAC to BC: Therefore as the compound
line BAC is to BC, ſo is BD to DE. Wherefore
the rectangle of the compound line BAC
and DE g is equal to the rectangle of BC and g 16. 6.
BD. But the rectangle of BC and BD is
given, (for that thoſe lines BC and BD are
given:) Therefore the rectangle under the compound
line BAC and ED is alſo given.

OTHERWISE.

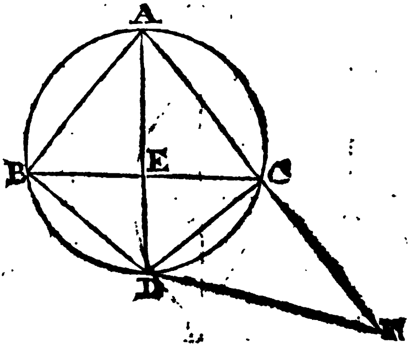


Conſtr. Let CA be prolonged to the point E, and let AE be put equal to BA, and let BE and BD be joined.

Demonſtr. Foraſmuch as the angle BAC is double to each of the angles CAD and AEB (for the angle BAC is cut into two equal parts by the line
h 32. 1. AD, and equal *h*
i 5. 1. to the two angles ABE and AEB, which *i* are
equal) the angle ABE is equal to the angle
k 21. 3. CAD, that is to ſay, *k* to the angle CBD; adding therefore the common angle ABC, the whole angle ABD ſhall be equal to the whole angle FBE. But the angle ACB is *k* equal to the angle ADB: Therefore the third angle AEB is equal to the third angle BAD; and therefore the triangle CEB is equiangled to the triangle ABD: Wherefore as CE is to CB, ſo is AD to BD. But the right line CE is compounded of the two lines CA and AB: Therefore as the compound line BAC is to CB, ſo is AD to BD; and by permutation, as the compound line BAC is to AD, ſo is CB to BD. But the reaſon of CB to BD is given, ſeeing that each of thoſe lines is given: Therefore the reaſon of the compound line BAC to AD is alſo given. And ſeeing that

the triangle CEB is equiangled to the triangle FBD (for the angle AFC is equal *l* to the an- l 21. 3.
gle BFD, and the angle ECB *m* to the angle m 16. 6.
ADB) as EC is to CB, ſo is BD to DF. But EC is equal to the compound line BAC: Therefore as the compound line BAC is to CB, ſo is BD to DF. Wherefore *n* the rect- n 16. 6.
angle of the compound line BAC and DF is equal to the rectangle of CB and BD. But the rectangle of CB and BD is given, conſidering that each of the lines CB and BD is given: Therefore the rectangle of the compound line BAC and DF is given.

OTHERWISE.

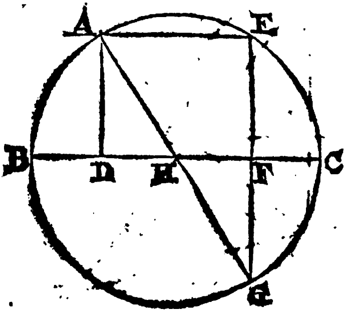


Conſtr. Let AC be prolonged to F, and let CF be put equal to AB, and let the right lines BD and DF be drawn.

Demonſtr. Foraſmuch as BA is equal to CF, and *o* BD to DC, the two ſides AB and BD o 26, 29.
are equal to the two ſides CD and DF, each 3.
to his correſponding ſide, and the angle ABD is equal to the angle DCF, *p* ſeeing that the p 22. 3.

four ſided figure ABDC is within the circle : q 4. 1. Therefore the baſe AD is *q* equal to the baſe DF, and the angle DAB to the angle DFC. But the angle BAD is given, (being the half of the given angle BAC) Therefore the angle DFC is ſo alſo. But DAF is alſo given : Therefore the triangle ADF is given by kind. Wherefore the reaſon of FA to AD is given. But AF is the compound of BA and AC, for that CF is equal to AB : Therefore the reaſon of the compound line BAC to AD is given: The ſame Demonſtration will ſerve to ſhew that the rectangle contained under the compound line BAC and ED is given alſo.

PROP. XCV.



If in the diameter BC of a circle ABC given by poſition, there be taken a given point D, and that from that point D there be drawn a right line DA, to the circumference of the circle. But from the ſection of the ſaid line there be drawn a right line AE, perpendicular thereto, and by the point E where that perpendicular doth meet with the circumference, there be drawn a parallel EF, to the firſt line drawn AD, that point F in which the parallel meets with the diameter, is given ; and the rectangle contained under the parallel lines AD and EF is alſo given.

Conſtr. Let the right line EF be prolonged to the point G, and let the right line AG be drawn.

Demonſtr. Foraſmuch as the angle AEG is a right angle, the right line AG is the diameter of the circle. But BC is alſo the diameter: Therefore the point H is the center of the circle. Now the point D is given;
and therefore *a* the line DH is given by mag- a 26.*prop.*
nitude. But ſeeing that AD is parallel to EG,
and AH equal to GH; *b* DH is equal to FH, b 26.*prop.*
and AD to FG; (for the angles AHD and
FHG *c* are equal, and DAH and FGH *d* are c 15. 1.
alſo equal.) But the line DH is given: d 29. 1.
Therefore FH is alſo given. But each of thoſe lines DH and HF is alſo given by poſition,
and the point H is given: Therefore *e* the e 27.*prop.*
point F is alſo given. And ſeeing that in the circle ABC given by poſition, is taken the given point F, and through the ſame is drawn the right line EFG; the rectangle un-
der EF and FG *f* is given. But FG is equal f 93.*prop.*
to AD. Therefore the rectangle comprehended under AD and EF is given. Which was to be demonſtrated.

The End of EUCLIDE's *DATA.*

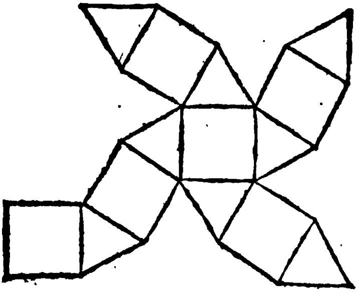
A
BRIEF TREATISE,
Added by *FLUSSAS*,
OF
Regular Solids.

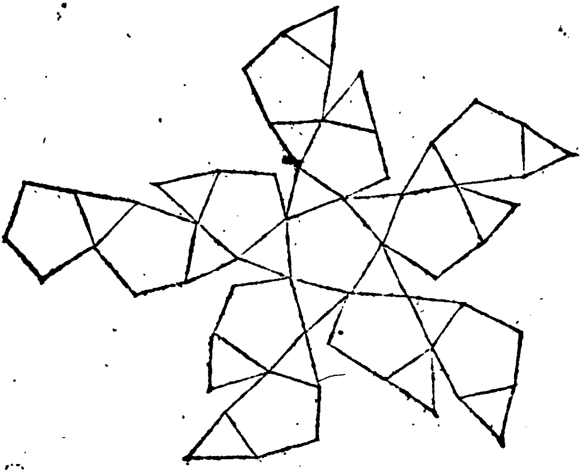
REgular Solids are ſaid to be compoſed and mix'd when each of them is transformed into other Solids, keeping ſtill the form, number and inclination of the baſes, which they before had to one another; ſome of which yet are transformed into mix'd Solids, and other ſome into ſimple.

Into mixt, as a Dodecaedron and an Icoſaedron, which are transformed or altered, if you divide their ſides into two equal parts, and take away the ſolid angles ſubtended of plain ſuperficial figures, made by the lines coupling thoſe middle ſections; for the Solid remaining after the taking away of thoſe ſolid angles, is called an Icoſidodecaedron. If you divide the ſides of a Cube and of an Octoedron into two equal parts, and couple the ſections, the ſolid angles ſubtended of the plain ſuperficies made by the coupling lines, being taken away, there ſhall be left a ſolid, which is called an Exoctoedron. So that both of a Dodecaedron and alſo of an Icoſaedron, the Solid which is made ſhall be called an Icoſidodecaedron; and likewiſe the Solid made of a Cube, and alſo of an Octoedron, ſhall be called an Exoctoedron. But the other Solid, to wit, a Pyramis or Tetraedron, is transformed into a ſimple Solid; for if you divide into two equal parts each of the ſides of the pyramis, triangles deſcribed of the lines which couple the ſections, and ſubtending and taking away the ſolid angles of the Pyramis, are equal and like unto the equilateral triangles left in each of the baſes, of all which triangles is produced an Octoedron, to wit, a ſimple, and not a compoſed Solid. For the Octoedron hath four baſes, like in number, form, and mutual inclination with the baſes of the pyramis, and hath the other four baſes with like ſituation oppoſite and parallel to the former. Wherefore the application of the pyramis taken twice, maketh a ſimple Octoedron, as the other Solids make a mix'd compound Solid.

DEFINITIONS.

I. *An Exoctoedron is a ſolid figure contained of ſix equal ſquares, and eight equilateral and equal triangles.*





II. *An Icoſidodecaedron is a ſolid figure contained under twelve equilateral, equal, and equiangled Pentagons, and twenty equal and equilateral triangles.*

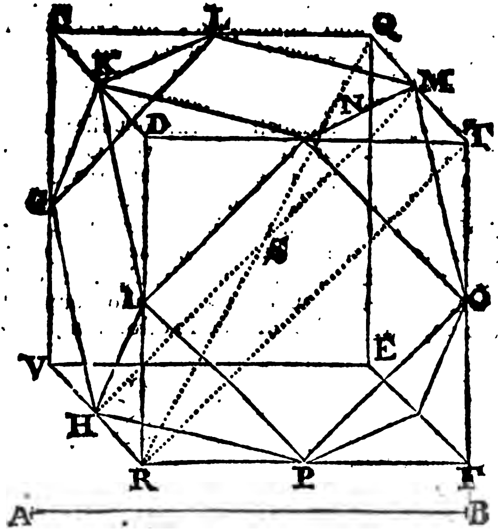
For the better Underſtanding of the two former Definitions, and alſo of the two Propoſitions following, I have here ſet two figures, whoſe figures if you firſt deſcribe upon paſted Paper or ſuch like matter, and then cut them and fold them accordingly, they will repreſent unto you the perfect forms of an Exoctoedron, and of an Icoſidodecaedron.

PROBLEME I.

To deſcribe an equilateral and equiangled Exoctoedron, and to contain it in a given Sphere, and to prove that the Diameter of the Sphere is double to the ſide of the ſaid Exoctoedron.

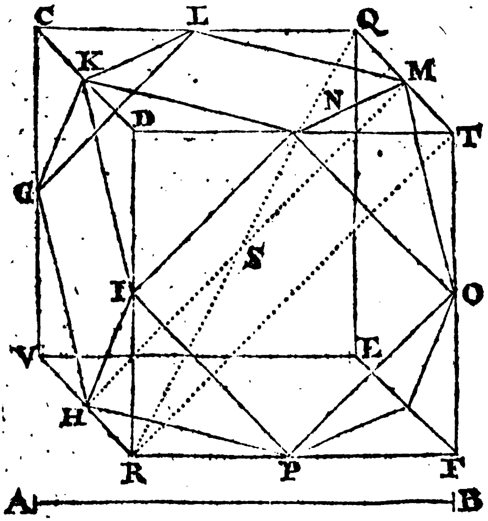
Conſtr. Suppoſe a Sphere whoſe diameter let be AB, and about the diameter AB let there
a 6. 4. be deſcribed a ſquare *a*, and upon the ſquare
b 15. 13. let there be deſcribed a Cube *b*, which let be CDEFQTVR; and let the diameter thereof be QR, and the center S. Divide the ſides of the Cube into two equal parts in the points G, H, I, K, L, M, N, O, P, *&c.* and couple the middle ſections by the right lines IN, NO, OP, PI, and ſuch like, which ſubtend the angles of the ſquares or baſes of the
c 4. 1. Cube; and they are equal *c*, and contain right angles, as the angle NIP. For the angle NID, which is at the baſe of the Iſoſceles triangle

triangle NDI, is the half of a right angle, and ſo likewiſe is the oppoſite angle RIP. Wherefore the reſidue NIP is a right angle, and ſo the reſt. Wherefore NIPO is a ſquare. And by the ſame reaſon ſhall the reſt NMLK, KGHI, &c. inſcribed in the baſes of the Cube, be ſquares, and they ſhall be ſix in number, according to the number of the baſes of the Cube. Again, foraſmuch as the triangle KIN ſubtendeth the ſolid angle D, of the Cube, and likewiſe the triangle KGL the ſolid angle C, and ſo the reſt which ſubtend the right ſolid angles of the Cube, and theſe triangles are equal and equilateral (to wit) being made of equal ſides, and they are the limits or borders of the ſquares, and the ſquares the limits or borders of them; as hath been



before proved. Wherefore LMNOPHGK is an Exoctoedron by the definition, and is equilateral; for it is contained of equal ſubtendant lines, it is alſo equiangled; for every ſolid angle thereof is contained under two ſuperficial angles of two ſquares, and two ſuperficial angles of two equilateral triangles.

Demonſtr. Foraſmuch as the oppoſite ſides and diameters of the baſes of the Cube are parallels, the plain extended by the right lines QT and VR, ſhall be a parallelogram. And for that alſo in that plain lyeth QR, the diameter of the Cube, and in the ſame plain alſo is the line MH, which divideth the ſaid plain into two equal parts, and alſo coupleth the oppoſite angles of the Exoctoedron; this



line MH therefore divideth the diameter into two equal parts d; and alſo divideth it ſelf d cor.34.1. in the ſame point, which let be S, into two equal parts e. e 4. 1. And by the ſame reaſon may we prove that the reſt of the lines which couple the oppoſite angles of the Exoctoedron, do in S the center of the Cube, divide one another into two equal parts, for each of the angles of the Exoctoedron are ſet in each of the baſes of the Cube. Wherefore making the center the point S, with the diſtance SH or SM deſcribe a Sphere, and it ſhall touch every one of the angles equidiſtant from the point S.

And foraſmuch as AB the diameter of the ſphere given, is put equal to the diameter of the baſe of the cube, to wit, to the line RT, and the ſame line RT is equal to the line MH *f*, which line MH coupling the oppoſite f 33. 1. angles of the Exoctoedron, is drawn by the center. Wherefore it is the diameter of the Sphere given which containeth the Exoctoedron.

Laſtly, foraſmuch as in the triangle RFT, the line PO doth cut the ſides into two equal parts, it ſhall cut them proportionally with the baſes, to wit, as FR is to FP, ſo ſhall RT be to PO *g*. But FR is double to g 2. 6. FP by ſuppoſition: Wherefore RT, or the diameter HM, is alſo double to the line PO, the ſide of the Exoctoedron. Wherefore we have deſcribed, *&c.* Which was required to be done.

PROBLEME II.

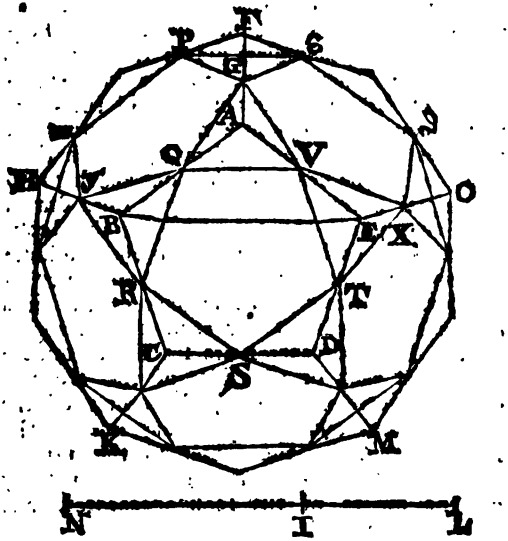
To deſcribe an equilateral and equiangled Icoſidodecaedron, and to comprehend it in a ſphere given, and to prove that the *diameter being divided by an extreme and mean proportion, maketh the greater ſegment double to the ſide of the Icoſi-dodecaedron.*

Conſtr. Suppoſe that the diameter of the
a 30. 6. ſphere given be NL, *a* divide the line NL,
by an extreme and mean proportion in the
point I, and the greater ſegment thereof let
be NI; and upon the line NI deſcribe a
b 15. 13. Cube *b*; and about this Cube let there be cir-
c 17. 13. cumſcribed a Dodecaedron *c*; and let the
ſame be ABCDEFHKMO, and divide each of
the ſides into two equal parts in the points
Q, R, S, T, V, X, Y, Z, P, e, a, G, &c.
and couple the ſections with right lines, which
ſhall ſubtend the angles of the Pentagons, as
the lines PG, GV, VQ, QY, YR, RQ, VT,
TX, XV, and ſo the reſt.

Demonſtr. Foraſmuch as theſe lines ſubtend
equal angles of the Pentagons, and thoſe equal
angles are contained of equal ſides, to wit, of
the halfs of the ſides of the Pentagons;
d 4. 1. therefore thoſe ſubtending lines are equal *d*.
Wherefore the triangles GQV, YQR, and
VXT, and the reſt, which take away ſolid
angles of the Dodecaedron, are equilateral.

Again, foraſmuch as in every Pentagon are deſcribed five equal right lines, coupling the middle ſections of the ſides, as are the lines QV, VT, TS, SR, and RQ, they deſcribe a Pentagon in the plain of the Pentagon of the Dodecaedron. And the ſaid Pentagon is contained in a circle, to wit, whoſe center is the center of a Pentagon of the Dodecaedron. For the lines drawn from that center to the angles of this Pentagon are equal, for that they are perpendiculars upon the baſes

cut a. Wherefore the Pentagon QRSTV, is e 12. 4.
equiangled f. And by the ſame reaſon may f 11. 4.



the reſt of the Pentagons deſcribed in the baſes of the Dodecaedron, be proved equal and like.

Wherefore thoſe Pentagons are twelve in number: And foraſmuch as the equal and like triangles do ſubtend and take away twenty ſolid angles of the Dodecaedron; therefore the ſaid triangles ſhall be twenty in number. Wherefore we have deſcribed an Icoſidodecaedron by the Definition, which Icoſidodecaedron is equilateral; for that all the ſides of the triangles are equal and common with the Pentagons; and it is alſo equiangled. For each of the ſolid angles is made of two ſuperficial angles of an equilateral Pentagon, and of two ſuperficial angles of an equilateral triangle.

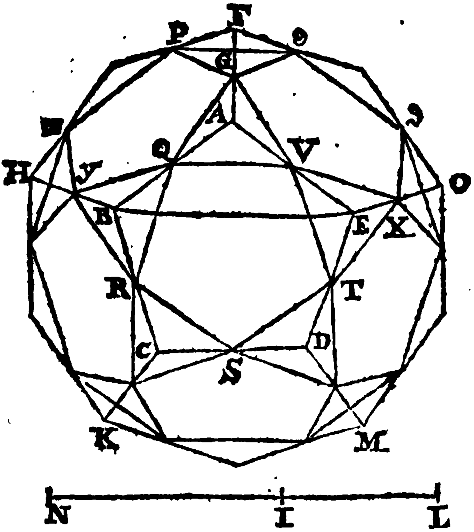
Now let us prove that it is contained in the given ſphere whoſe diameter is NL. Foraſmuch as perpendiculars drawn from the centers of the Dodecaedron, to the middle ſections of his ſides, are the halfs of the lines which couple the oppoſite middle ſections of the ſides of the Dodecaedron g; which lines alſo h do in the center divide one another into two equal parts. Therefore right lines drawn from that point to the angles of the Icoſidodecaedron (which are ſet in thoſe middle ſections) are equal; which lines are thirty in number, according to the number of the ſides of the Dodecaedron; for each of the angles of the Icoſidodecaedron are ſet in the middle ſections of each of the ſides of the Dodecaedron. Wherefore making the center of the Dodecaedron, and the ſpace any one of the lines drawn from the center to the middle ſections, deſcribe a ſphere, and it ſhall paſs by all the angles of the Icoſidodecaedron, and ſhall contain it.

g 3 *cor.* 17. 13.
h *idem.*

And foraſmuch as the diameter of this ſolid, is that right line whoſe greater ſegment is the ſide of the Cube inſcribed in the Dodecaedron i, which ſide is NI by ſuppoſition. Wherefore that ſolid is contained in the ſphere given, whoſe diameter is put to be the line NL.

i 4. *cor.* 17. 13.

Now



Now let us prove that the greater ſegment of the diameter is double to QV the ſide of the ſolid. Foraſmuch as the ſides of the triangle AEB, are in the points Q and V divided into two equal parts, the lines QV and BE are parallels *k*. Wherefore as AE is to AV, ſo is EB to VQ *l*. But the line AE is double to the line AV. Wherefore the line BE is double to the line QV *m*. Now the line BE is equal to NI, or to the ſide of the Cube *n*; which line NI is the greater ſegment of the diameter NL. Wherefore the greater ſegment of the diameter given is double to the ſide of the Icoſidodecaedron inſcribed in the given ſphere. Wherefore, We have deſcribed, &c. Which was required to be done.

k *cor.* 39. 1.
l 2. 6.
m 4. 6.
n 2 *cor.* of 17. 13.

ADVERTISEMENT.

To the Underſtanding of the nature of this Icoſidodecaedron, you muſt well conceive the paſſions and proprieties of both theſe ſolids, of whoſe baſes it conſiſteth, to wit, of the Icoſaedron and of the Dodecaedron. And altho' in it the baſes are placed oppoſitely, yet have they to one another one and the ſame inclination. By reaſon whereof there lye hidden in it the actions and paſſions of the other Regular Solids. And I would have thought it not impertinent to the purpoſe to have ſet forth the inſcriptions and circumſcriptions of this Solid, if want of time had not hindred. But to the end the Reader may the better attain to the Underſtanding thereof, I have here following briefly ſet forth, how it may in or about every one of the five Regular Solids be inſcribed or circumſcribed; by the help whereof he may, with ſmall travel or rather none at all, having well poiſed and conſidered the Demonſtrations appertaining to the foreſaid five Regular Solids, demonſtrate both the inſcription of the ſaid Solids in it, and the inſcription of it in the ſaid Solids.

Of the Inſcriptions and Circumſcriptions of an Icoſidodecaedron.

An Icoſidodecaedron may contain the other five regular bodies. For it will receive the angles of a Dodecaedron in the centers of the triangles which ſubtend the ſolid angles of the Dodecaedron, which ſolid angles are twenty in number, and are placed in the ſame order in which the ſolid angles of the Dodecaedron taken away, or ſubtended by them, are. And by that reaſon it ſhall receive a Cube and a

Pyramis contained in the Dodecaedron, when as the angles of the one, are ſet in the angles of the other.

An Icoſidodecaedron receiveth an Octoedron, in the angles cutting the ſix oppoſite ſections of the Dodecaedron, even as if it were a ſimple Dodecaedron.

And it containeth an Icoſaedron, placing the twelve angles of the Icoſaedron in the ſame centers of the twelve Pentagons.

It may alſo by the ſame reaſon be inſcribed in each of the five regular bodies, to wit, in a Pyramis, if you place four triangular baſes concentrical with four baſes of the Pyramis, after the ſame manner that you inſcribed an Icoſaedron in a Pyramis; ſo likewiſe may it be inſcribed in an Octoedron, if you make eight baſes thereof concentrical with the eight baſes of the Octoedron. It ſhall alſo be inſcribed in a Cube, if you place the angles which receive the Octoedron in it, in the centers of the baſes of the Cube. Again, you ſhall inſcribe it in an Icoſaedron, when the triangles compaſſed in of the Pentagon baſes, are concentrical with the triangles which make a ſolid angle of the Icoſaedron.

Laſtly, It ſhall be inſcribed in a Dodecaedron, if you place each of the angles thereof in the middle ſections of the ſides of the Dodecaedron, according to the order of the Conſtruction thereof.

The oppoſite plain ſuperficies alſo of this ſolid are parallels. For the oppoſite ſolid angles are ſubtended of parallel plain ſuperficies, as well in the angles of the Dodecaedron ſubtended by triangles, as in the angles of the Icoſaedron ſubtended of Pentagons, which thing may eaſily be demonſtrated. Moreover, in this ſolid are infinite properties and paſſions, ſpringing of the ſolids whereof it is compoſed.

Wherefore it is manifeſt, that a Dodecaedron and an Icoſaedron mixed, are transformed into one and the ſelf ſame ſolid of an Icoſidodecaedron. A Cube alſo and an Octoedron are mixed and altered into another ſolid, to wit, into one and the ſame Exoctoedron. But a Pyramis is transformed into a ſimple and perfect ſolid, to wit, into an Octoedron.

If we will frame theſe two ſolids joined together into one ſolid, this only muſt we obſerve.

In the Pentagon of a Dodecaedron inſcribe a like Pentagon, and let its angles be ſet in the middle ſections of the Pentagon circumſcribed, and then upon the ſaid Pentagon inſcribed, let there be ſet a ſolid angle of an Icoſaedron, and ſo obſerve the ſame order in each of the baſes of the Dodecaedron, and the ſolid angles of the Icoſaedron ſet upon theſe Pentagons ſhall poduce a ſolid conſiſting of the whole Dodecaedron, and whole Icoſaedron. In like ſort, if in every baſe of the Icoſaedron, the ſides being divided into two equal parts, be inſcribed an equilateral triangle, and upon each of thoſe equilateral triangles be ſet a ſolid angle of a Dodecaedron, there ſhall be produced the ſame ſolid conſiſting of the whole Icoſaedron, and of the whole Dodecaedron.

And after the ſame order, if in the baſes of a Cube be inſcribed ſquares ſubtending the ſolid angles of an Octoedron, or in the baſes of an Octoedron be inſcribed equilateral triangles ſubtending the ſolid angles of a Cube, there ſhall be produced a ſolid conſiſting of either of the whole ſolids, to wit, of the whole cube, and of the whole Octoedron.

But equilateral triangles inſcribed in the baſes of a Pyramis, having their angles ſet in the middle ſections of the ſides of the Pyramis, and the ſolid angles of a Pyramis, ſet upon the ſaid equilateral triangles, there ſhall be produced a ſolid conſiſting of two equal and like pyramids.

And now if in theſe ſolids thus compoſed, you take away the ſolid angles, there ſhall be reſtored again the firſt compoſed ſolids, to wit, the ſolid angles taken away from a Dodecaedron and an Icoſaedron compoſed into one, there ſhall be left an Icoſidodecaedron, the ſolid angles taken away from a Cube and an Octoedron compoſed into one ſolid, there ſhall be left an Exoctoedron. Moreover, the ſolid angles taken away from two pyramids compoſed into one ſolid, there ſhall be left an Octoedron.

Of the nature of a trilateral and equilateral Pyramis.

1. A trilateral equilateral Pyramis is divided into two equal parts, by three equal ſquares, which in the center of the Pyramis cut one another into two equal parts, and perpendicularly, and whoſe angles are ſet in the middle ſections of the ſides of the Pyramis.

2. From a Pyramis are taken away four Pyramids like unto the whole, which utterly take away the ſides of the Pyramis, and that which is left is an Octoedron inſcribed in the Pyramis, in which all the ſolids inſcribed in the Pyramis are contained.

3. A perpendicular drawn from the angle of the Pyramis to the baſe, is double to the diameter of the Cube inſcribed in it.

4. And a right line coupling the middle ſections of the oppoſite ſides of the Pyramis is triple to the ſide of the ſame Cube.

5. The ſide alſo of a Pyramis is triple to the diameter of the baſe of the Cube.

6. Wherefore the ſame ſide of the Pyramis is in power double to the right line which coupleth the middle ſections of the oppoſite ſides.

7. And it is in power ſeſquialter to the perpendicular which is drawn from the angle to the baſe.

8. Wherefore the perpendicular is in power ſeſquitertia to the line which coupleth the middle ſections of the oppoſite ſides.

9. A Pyramis and an Octoedron inſcribed in it, alſo an Icoſaedron inſcribed in the ſame Octoedron, do contain one and the ſame ſphere.

Of the nature of an Octoedron.

1. Four perpendiculars of an Octoedron, drawn in four baſes thereof from two oppoſite angles of the ſaid Octoedron, and coupled together by thoſe four baſes, deſcribe a Rhombus, or Diamond figure; one of whoſe diameters is in power double to the other diameter.

2. For it hath the ſame proportion that the diameter of the Octoedron hath to the ſide of the Octoedron.

3. An Octoedron and an Icoſaedron inſcribed in it, do contain one and the ſame ſphere.

4. The diameter of the ſolid of the Octoedron is in power ſeſquialter to the diameter of the circle which containeth the baſe, and is in power duple ſuperbipartiens tertias (that is, as 8 to 3,) to the perpendicular or ſide of the foreſaid Rhombus; and moreover

is in length triple to the line which coupleth the centers of the next baſes.

5. The angle of the inclination of the baſes of the Octoedron, doth with the angle of the inclination of the baſes of the Pyramis, make angles equal to two right angles.

Of the nature of a Cube.

1. The diameter of a Cube is in power ſeſquialter to the diameter of his baſe.

2. And is in power triple to his ſide.

3. And unto the line which coupleth the centers of the next baſes, it is in power ſextuple.

4. Again, the ſide of the Cube, is to the ſide of the Icoſaedron inſcribed in it, as the whole is to the greater ſegment.

5. Unto the ſide of the Dodecaedron, it is as the whole is to the leſſer ſegment.

6. Unto the ſide of the Octoedron it is in power duple.

7. Unto the ſide of the Pyramis it is in power ſubduple.

8. Again, the Cube is triple to the Pyramis, but to the cube the Dodecaedron is in a manner double. Wherefore the ſame Dodecaedron is in a manner ſextuple to the ſaid Pyramis.

Of the nature of the Icoſaedron.

1. Five triangles of an Icoſaedron, do make a ſolid angle, the baſes of which triangles make a Pentagon. If therefore from the oppoſite baſes of the Icoſaedron be taken the other Pentagon by them deſcribed, theſe Pentagons ſhall in ſuch ſort cut the diameter of the Icoſaedron which coupleth the foreſaid oppoſite angles,

that

that that part which is contained between the planes of theſe two Pentagons ſhall be the greater ſegment, and the reſidue, which is drawn from the plain to the angle, ſhall be the leſſer ſegment.

2. If the oppoſite angles of two baſes joined together, be coupled by a right line, the greater ſegment of that right line is the ſide of the Icoſaedron.

3. A line drawn from the center of the Icoſaedron to the angles, is in power quintuple to half that line, which is taken between the Pentagons, or of the half of that line, which is drawn from the center of the circle which containeth the foreſaid Pentagon, which two lines are therefore equal.

4. The ſide of the Icoſaedron containeth in power either of them, and alſo the leſſer ſegment, to wit, the line which falleth from the ſolid angle to the Pentagon.

5. The diameter of the Icoſaedron containeth in power the whole line, which coupleth the oppoſite angles of the baſes joined together, and the greater ſegment thereof, to wit, the ſide of the Icoſaedron.

6. The diameter alſo is in power quintuple to the line which was taken between the Pentagons, or to the line which is drawn from the center to the circumference of the circle which containeth the Pentagon compoſed of the ſides of the Icoſaedron.

7. The dimetient containeth in power the right line which coupleth the centers of the oppoſite baſes of the Icoſaedron, and the diameter of the circle which containeth the baſe.

8. Again, the ſaid dimetient containeth in power the diameter of the circle which containeth the Pentagon, and alſo the line which is drawn from the center of the ſame circle to the circumference; that is, it is quintuple

to the line drawn from the center to the circumference.

9. The line which coupleth the centers of the oppoſite baſes, containeth in power the line which coupleth the centers of the next baſes, and alſo the reſt of that line of which the ſide of the Cube inſcribed in the Icoſaedron is the greater ſegment.

10. The line which coupleth the middle ſections of the oppoſite ſides, is triple to the ſide of the Dodecaedron inſcribed in it.

11. Wherefore if the ſide of the Icoſaedron, and the greater ſegment thereof be made one line, the third part of the whole is the ſide of the Dodecaedron inſcribed in the Icoſaedron.

Of the Dodecaedron.

1. The diameter of a Dodecaedron containeth in power the ſide of the Dodecaedron, and alſo that right line to which the ſide of the Dodecaedron is the leſſer ſegment, and the ſide of the Cube inſcribed in it is the greater ſegment, which line is that which ſubtendeth the angle of the inclination of the baſes, contained under two perpendiculars of the baſes of the Dodecaedron.

2. If there be taken two baſes of the Dodecaedron, diſtant from one another by the length of one of the ſides, a right line coupling their centers being divided by an extreme and mean proportion, maketh the greater ſegment the right line which coupleth the centers of the next baſes.

3. If by the centers of five baſes ſet upon one baſe, be drawn a plain ſuperficies, and by the centers of the baſes which are ſet upon the oppoſite baſe, be drawn alſo a plain ſuperficies, and then be drawn a right line, coupling the centers of the oppoſite baſes,

that right line is ſo cut, that each of his parts ſet without the plain ſuperficies, is the greater ſegment of that part which is contained between the plains.

4. The ſide of the Dodecaedron is the greater ſegment of the line which ſubtendeth the angle of the Pentagon.

5. A perpendicular line drawn from the center of the Dodecaedron to one of the baſes, is in power quintuple to half the line which is between the plains.

6. And therefore the whole line which coupleth the centers of the oppoſite baſes, is in power quintuple to the whole line which is between the ſaid plains.

7. The line which ſubtendeth the angle of the baſe of the Dodecaedron, together with the ſide of the baſe are in power quintuple to the line which is drawn from the center of the circle which containeth the baſe, to the circumference.

8. A ſection of a ſphere containing three baſes of the Dodecaedron, taketh a third part of the diameter of the ſaid ſphere.

9. The ſide of the Dodecaedron and the line which ſubtendeth the angle of the Pentagon, are equal to the right line which coupleth the middle ſections of the oppoſite ſides of the Dodecaedron.

THE
THEOREMS
OF
ARCHIMEDES.

Concerning the Sphere and Cylinder, Inveſtigated by the *Method of Indiviſibles*, and briefly Demonſtrated by the Reverend and Learned Dr. *Iſaac Barrow*.

THE main Deſign of *Archimedes* in his *Treatiſe of the Sphere and Cylinder*, is to reſolve theſe four Problems.

1. *To find the proportion of the ſuperficies of a ſphere to any determinate circle; or to find a circle equal to the ſuperficies of a given ſphere.*

2. *To find the proportion of the ſuperficies of any ſegment of a ſphere to any determined circle; or to find a circle equal to the ſuperficies of any aſſign'd ſegment.*

3. *To find the proportion of the ſphere it ſelf (or of its ſolid content) to any determinate Cone or Cylinder; or to find a Cone or Cylinder equal to a given ſphere.*

4. *To find the proportion of a ſegment of a ſphere to any determinate Cone or Cylinder; or to find a Cone or Cylinder equal to a given ſegment.*

Theſe four Problems *Archimedes* proſecutes ſeparately, and lays down Theorems immediately ſubſervient to their ſolution; but we reduce them to two: For ſince an Hemiſphere is the ſegment of a ſphere, and the method of finding out its relations, in reſpect to the ſuperficies and ſolid content, is comprehended in the general method of inveſtigating the proportion of the ſegments: And from the ſuperficies and ſolid content of an Hemiſphere already found, the double of them, (that is, the ſuperficies and content of the whole ſphere) is at the ſame time given. And indeed 'tis ſuperfluous and foreign from the Laws of good Method, to inveſtigate their relations diſtinctly and ſeparately; ſo that if it were not a crime, I might on this account, blame even *Archimedes* himſelf.

The whole matter therefore is reduc'd to theſe two Problems.

1. *To find the proportion of the ſuperficies of any ſegment of a ſphere, to a determinate circle; or to find a circle equal to the ſuperficies of a given ſegment.*

2. *To find the proportion of the ſolidity of any ſegment of a ſphere to any determinate Cone or Cylinder; or to find a Cone or Cylinder equal to an aſſign'd ſegment of a ſphere.*

I ſhall reſolve theſe Problems by another much eaſier and ſhorter method: In which the order being inverted, firſt, I ſhall ſeek the ſolidity of a ſegment, and from thence deduce its ſuperficies; a thing which is in my judgment well worth obſerving, and perform'd, as I know of, by none.

Firſt therefore, for finding the ſolidity of a ſegment, I ſhall lay down two, commonly known and receiv'd Suppoſitions, *viz.*

1. *That a ſeries of magnitudes proceeding in Arithmetical Progreſſion from nothing (incluſive) or whoſe common difference is equal to the leaſt magnitude, is ſubduple of as many quantities equal to the greateſt*; (i. e. *ſubduple of the product of the greateſt term and number of terms:*) So that if the *ſum of the terms* be called *z*, the *greateſt term g*, and the *number of terms n*, then will

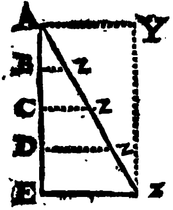
$$z = \frac{ng}{2}, \text{ or } 2z = ng.$$

The truth of this Propoſition will eaſily appear by expreſſing the ſeries twice, and inverting the order;

$$\begin{matrix} 0, & a, & 2a, & 3a, & 4a. \\ 4a, & 3a, & 2a, & a, & 0. \end{matrix}$$

For ſo the difference always being equal to the leaſt quantity, 'twill be evident that each two correſpondent terms taken together are equal to the greateſt term; and alſo, that the ſeries taken twice is equal to the greateſt term repeated as many times as there are terms, *i. e.* the laſt term drawn into the number of terms.

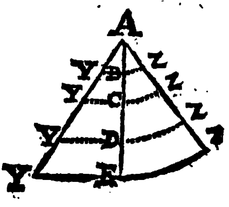
We have in a *triangle* a very clear and eaſy example of this moſt uſeful Propoſition, which is prov'd hence, *to be half a parallelogram having the ſame altitude, and ſtanding on the ſame baſe.*



Suppoſe the altitude AE of the triangle AEZ to be divided into parts indefinitely many and ſmall AB, BC, CD, DE, and parallels BZ, CZ, DZ, EZ, drawn thro' the points of Diviſions; all theſe proceed from *o* in an Arithmetical Progreſſion, and conſequently the ſum of 'em all, that is, the triangle AEZ, is ſubduple of the greateſt EZ drawn into the altitude AE, by which the ſum of the terms is expreſs'd, that is, ſubduple of the Parallelogram EY, whoſe baſe is EZ, and altitude AE.

But the illuſtration of the Rule will conduce more to our deſign by inferring hence, *That a circle is equal to half of the radius drawn into the circumference*, after this manner. Conceive a circle to conſiſt of as many concentric Peripheries as there are points or equal parts indefinitely many and ſmall in the radius. Theſe Peripheries, as well as their radii proceed from the center or nothing in an Arithmetical progreſſion; and therefore their ſum, that is, the whole circle is equal to half the greateſt (or extreme circumference) drawn into the number of terms, that is, the radius.

After the ſame manner we may ſuppoſe the ſector AEZ to conſiſt of as many concentric Arcs BZ, CZ, DZ, EZ, as there are points (or equal parts indefinitely ſmall) in the radius AE, which Arcs, as their radii, proceeding from a point or nothing in an Arithmetical progreſſion, the ſector



alſo will be equal to half the radius drawn into the extreme Arc EZ. Which may be made evident alſo after this manner: Let us ſuppoſe the right line EY to be perpendicular to the radius AE, draw the right line AY, and from the points B, C, D, of diviſion in the radius, draw BY, CY, DY, parallel to EY, and terminated at AY. Becauſe EY : DY (:: *rad.* AE : *rad.* AD) :: *Arc* EZ : *Arc* DZ. and $EY = EZ$, then will $DY =$ Arc DZ; and in like manner will $CY = CZ$, and $BY = BZ$. whence the triangle AEY will be = to the ſector AEZ, that is, $\frac{AE \times EY}{2}$ ($\frac{AE \times EZ}{2}$) = ſector AEZ. By this means we collect that celebrated Theorem of *Archimedes*, *That a circle is equal to a triangle, whoſe baſe is equal to the radius, and altitude equal to the periphery of the circle*; and that without any inſcription or circumſcription, of figures, by only ſuppoſing that the Area or Superficies of the circle conſiſts of infinitely many concentric Peripheries. Which *method of indiviſibles*, (now firſt of all known to me) ſeems no leſs evident (nay more evident) and perhaps leſs fallacious than that wherein planes are ſuppoſed to conſiſt of parallel right lines, and ſolids of parallel planes; as hereafter ſhall be evident, when we ſhall collect, by this method, the proportions of ſpheric and cylindric ſuperficies to one another, by knowing the ſolid content; and on the other hand, the ſolid content, by knowing the ſuperficies, with admirable facility, and moſt full ſatisfaction in thoſe things which are rigidly gather'd by pure *Geometry*.

2. *Let us ſuppoſe a ſeries of quantities to proceed from* 0 (*incluſive*) *in a duplicate Arithmetic progreſſion, that is,* 0, 1, 4, 9, 16, &c. *the ſquares*

of numbers in a ſimple Arithmetic progreſſion, 0, 1, 2, 3, 4, &c. *And the triple of this ſeries will always exceed the greateſt term multiplied by the number of terms; but the number of terms increaſing the proportion, continually approximates till at laſt it comes to an equality, when the number of terms is increaſed* in infinitum.

$$\begin{array}{r} 3\times 0+1=3.\\ 2\times 1=2. \end{array}\;\frac{3}{2}$$

$$\begin{array}{r} 3\times 0+1+4=15.\\ 3\times 4=12. \end{array}\;\frac{15}{12}=\frac{5}{4}$$

$$\begin{array}{r} 3\times 0+1+4+9=42.\\ 4\times 9=36. \end{array}\;\frac{42}{36}=\frac{7}{6}$$

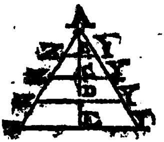
$$\begin{array}{r} 3\times 0+1+4+9+16=90.\\ 5\times 16=80. \end{array}\;\frac{90}{80}=\frac{9}{8}$$

$$\begin{array}{r} 3\times 0+1+4+9+16+25=165.\\ 6\times 25=125. \end{array}\;\frac{165}{125}=\frac{11}{10}$$

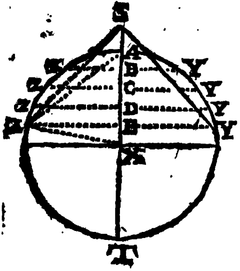
As for example, if the terms be two, the triple of the terms will be to the greateſt term drawn into the number of terms as 3 to 2; if there be three terms as 5 to 4; if four, as 7 to 6; if five, as 9 to 8, and ſo continually: So that the antecedents of theſe proportions always mutually exceed one another by the number 2; and ſo every antecedent its conſequent by 1. Whence it is evident that by how much the greater the number of terms is, by ſo much the more the proportion tends to equality. So 100 to 99 is leſs diſtant from the proportion of equality than 10 to 9. From hence, ſuppoſing the number of terms infinite (or infinitely great,) the triple of quantities proceeding thus in a duplicate proportion (or as the ſquares of the numbers, 0, 1, 2, 3, 4, *&c.* will be equal to as many quantities equal to the greateſt term.

The ſame, as to the ſubſtance of it, is laid down by *Archimedes* in his *Book of Spirals*, as the Foundation of many Argumentations, in that, and other Books, and is well demonſtrated by our Learned Country-man Dr. *Wallis*: However, I thought fit to illuſtrate the matter by this method, as being not unworthy our Conſideration, and very perſpicuous and intelligible in this, that 'tis free from Fractions. And by the way 'tis obſerv'd, that from hence we may eaſily find the proportion of a ſeries triple to as many terms equal to the greateſt, *viz.* as twice the number of terms leſs one, to twice the number of terms leſs two. So that if the number of terms be 6, the proportion of a ſeries triple to as many terms equal to the greateſt will be as 11.

It will be a very eaſy and apt Illuſtration of this Rule, if we infer hence, *That a Cone is ſubtriple of a Cylinder, having an equal baſe and altitude.* For let us ſuppoſe the altitude AE of the cone ZY to be divided into equal and indefinitely many parts, by as many parallel right lines ZY, and the lines ZY will be as the numbers 1, 2, 3, 4, *&c.* and the ſquares or circles conſtituted upon the diameters ZY, as 1, 4, 9, 16, *&c.* whence all thoſe circles, or the whole cone AZY (made up of the ſame) will be ſubtriple of as many circles equal to the greateſt, conſtituted on the greateſt diameter ZEY, that is, ſubtriple of a cylinder whoſe baſe is AEY, and altitude AE.



There occur two other moſt apt examples of this Rule, *viz.* by inferring, *That the complement of a Semiparabola is ſubtriple of a parallelogram having the ſame baſe and height*; as alſo, *That the ſpace comprehended by the Spiral and Radius is ſubtriple of the circle in which the Spiral is generated:* But of theſe in another place. Wherefore to go on with what we began, theſe two Rules being ſuppoſed; let us conceive ZAY to be a ſegment of a ſphere, X its center, AT its diameter, and ZAYT a great circle paſſing thro' the vertex, and the part AE of the Axe to be divided into an indefinitely many equal parts; and let us imagine parallel lines to be drawn thro' the points of Diviſion, generating circles in the ſphere, whoſe Radii let be BZ, CZ, DZ, and diameters ZY. I ſuppoſe the ſegment of a ſphere to conſiſt of all theſe parallel circles, whoſe number is as great as that of the points, or equal indefinitely many ſmall parts in the Axe AE, according to the known *Method of Indiviſibles.*



But now for brevity's ſake, let the diameter AT be called d, and the radius of the ſphere r (if need be,) and the Axe AE, by which the number of terms is expreſs'd, call n, and one of the equal parts a; which being ſettled, 'tis evident, (*by the Elements*) that $\overline{BZ}^2 = AB \times BT = a \times \overline{d-a} = ad - a^2$, and in like manner $\overline{CZ}^2 = AC \times CT = 2a \times \overline{d-2a} = 2ad - 4a^2$, and by the ſame reaſoning $\overline{DZ}^2 = AD \times DT = 3ad - 9a^2$, and $\overline{EZ}^2 = AE \times ET = 4ad - 16a^2$, &c. that is, that the ſquares of the radii of the circles ZY are to one another as the rectangles ad, $2ad$, $3ad$, $4ad$, &c. (which proceed in an Arithmetical Progreſſion from o) leſs by the ſquare a^2, $4a^2$, $9a^2$, $16a^2$, &c. which go on as the ſquares of the numbers, 1, 2, 3, 4, &c.) But by our firſt Rule, all the Rectangles o, ad. $2ad$, $3ad$, $4ad$, &c. are equal to half as many terms equal to the greateſt AE × AT or nd, that is, $= \frac{nd \times n}{2}$.

Moreover, by our ſecond Rule, all the ſquares o, a^2, $4a^2$, $9a^2$, $16a^2$, &c. taken together, are equal to a third part of as many terms equal to the greateſt $\overline{AE}^2$ or n^2, that is, $= \frac{n^2 \times n}{3}$.

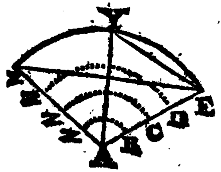
Wherefore all the ſquares deſcribed upon the radii BZ, CZ, DZ, EZ, conjunctly, are equal to the difference $\frac{ndn}{2}\ \frac{nnn}{3}$, (or the terms being reduc'd to the ſame denomination,) $\frac{3\,ndn - 2\,nnn}{6}$ and their quadruple, that is, all the ſquares deſcribed upon the diameter ZY, are equal to $\frac{12\,ndn - 8\,n^3}{6}$ or $\frac{6\,ndn - 4\,n^3}{3}$. Whence a ſegment of a ſphere is equal to a Cylinder, the diameter of whoſe baſe is the ſide of a ſquare equal to $6\,nd - 4\,n^2$, and altitude is $\frac{1}{3}n$; or to a cone having the ſame baſe, but the altitude n, or which is all one, having a baſe whoſe radius is $\sqrt{\frac{6\,nd - 4\,n^2}{4}}$ or $\sqrt{\frac{3}{2}\,nd - n^2}$, and altitude n as before. Which Cone we may change into a Cone upon the ſame baſe ZY with the ſegment ZAY, by ſaying, as ZE^2 (*i. e.* $dn - n^2$) to $\frac{3}{2}\,nd - n^2$ or (both terms being divided by n) as $d - n$ to $\frac{1}{2}d - n$, ſo reciprocally n to the altitude of the Cone ſought: Or in the figure by making, *as TE to TE + XA, ſo is EA to ES. For ES will be the altitude of the Cone ZSY equal to the ſegment of the ſphere ZAY.* Which is a noted Theorem of *Archimedes*, demonſtrated by him with ſo much labour and prolixity.

Hence, if the given ſegment be a *Hemiſphere*, and ſo $n = \frac{1}{2}d$ or r, then d or $2r$ will be the altitude of a *Cone*, which having a baſe equal to the baſe of the *Hemiſphere*, (or to the greateſt circle in the ſphere,) will be equal to the *Hemiſphere*. And a *Cone* whoſe baſe is double of the greateſt circle, and the altitude $2r$, or the *Cylinder* whoſe baſe is $\frac{2}{3}$ of the greateſt circle, and altitude $2r$, will be equal to the whole *Sphere*. Whence the whole *Sphere* is $\frac{2}{3}$ of a *Cylinder* the diameter of whoſe baſe is $2r$, and the altitude alſo $2r$. And this is the chief Theorem of *Archimedes*, viz. *That a ſphere is ſubſeſquialter or $\frac{2}{3}$ of that Cylinder, whoſe Altitude and Diameter of the baſe is equal to the Diameter of the Sphere.*

Furthermore, not to paſs over any thing in our Author which ſeems to be to our purpoſe:

If to the ſum firſt found, repreſenting a ſegment, *viz.* $\frac{6ndn - 4nnn}{3}$, we add $\frac{2d^2n - 6dn^2 + 4n^3}{3}$ $\left(= \frac{4dn - 4n^2}{3} \times \frac{d - 2n}{2} = \frac{4}{3}\overline{ZE}^2 \times XE\right)$ repreſenting the Cone ZXY, the aggregate $\frac{2}{3}ddn$ will repreſent the Sector of the Sphere ZXYA, which for that reaſon will be equal to a Cylinder, the diameter of whoſe baſe $\sqrt{dn}$, and the altitude $\frac{2}{3}d$, or to a Cone, the diameter of whoſe baſe is $\sqrt{dn}$, and the altitude $2d$, or alſo in a Cone, the Radius of whoſe baſe is $\sqrt{dn}$, and the altitude $\frac{1}{2}d = r$ (it being reciprocally as $4dn : dn :: 2d : \frac{1}{2}d$,) that is, to a Cone, the Radius of whoſe baſe is the Line AZ, drawn from the vertex to the circumference of the baſe of the ſegment, (for $\overline{AZ}^2 = TA \times AE = dn$,) and the altitude r. And this is the next famous Theorem of *Archimedes*, concerning the ſolidity of the ſector of the Sphere, *viz. That the ſector of a ſphere is equal to a Cone, whoſe baſe is a circle deſcribed by a Radius equal to a line drawn from the vertex to the circumference of the baſe of the ſegment, and whoſe altitude is equal to the Radius of the ſphere.*

And thus I think I have completed that which belongs to the ſolidity of a ſphere, and its parts with ſufficient brevity and perſpicuity. From hence we ſhall deduce the Reſolution of the other Problem, which I propoſed concerning the ſurface of the ſegment of a ſphere; and then of the whole ſphere. To obtain this, as we ſuppoſed, before a *Circle* to conſiſt of concentric Peripheries, and the *Sector of a Circle*, of concentric Arcs, (in the number of which, the greateſt, and the leaſt or a point is reckon'd: So now we ſuppoſe ſpheres to conſiſt of concentric ſpherical ſuperficies, and the *Sectors of Spheres*, of like concentric ſuperficies; as for example, the ſector of the ſphere ZAE, of the ſuperficies BZ, CZ, DZ, EZ, &c.) which ſuppoſition indeed ſeems ſo eaſy and natural, that in my judgment 'tis ſufficient



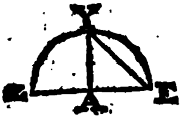
cient only to propoſe it; neither is a further explication wanting to gain an aſſent to it.

2. We ſuppoſe theſe ſpherical ſuperficies to be in a duplicate Ratio of the Radius of the ſpheres: This is the common affection of all like ſuperficies, and it ſeems to agree very well with the ſuperficies of ſpheres, becauſe they appear to be moſt uniform and ſimilar. But this Suppoſition might eaſily be evinc'd and eſtabliſh'd by the ſame ſort of arguing, as ſpheres are proved to be in triplicate proportion to their Diameters or Radii; Or might have been join'd as a *Corollary* to *Prop*. 17. and 18. *Elem*. 12. where the ſuperficies of like Polygones are ſuppos'd to be inſcribed in ſpheres, having as well the ſuperficies in a duplicate, as the ſolidity in a triplicate Ratio of the Diameters of the Spheres. Theſe things being premis'd, let us ſuppoſe AE a Radius or the ſide of the Sector of a Sphere EAZ, to be divided into equal and indefinitely many ſmall parts, and the ſector AEZ to conſiſt of theſe ſpherical ſuperficies BZ, CZ, DZ, EZ, it will be evident that all thoſe ſuperficies in the Progreſſion are as the ſquares of the Radii, that is, as $\overline{AB}^2$, $\overline{AC}^2$, $\overline{AD}^2$, $\overline{AE}^2$, *&c.* or as the ſquares of the numbers 1, 2, 3, 4, *&c.* whence by our ſecond Rule, the ſum of all theſe ſuperficies, that is, the ſector AEZ, will be $\frac{1}{3}$ of as many ſuperficies equal to the greateſt FZ, that is, $\frac{1}{3}$ of the greateſt EZ, drawn into r the number of terms. Whence a ſector is equal to a Cylinder whoſe baſe is $\frac{1}{3}$ of the greateſt or extreme ſuperficies of the ſector, and whoſe altitude is r: Or to a Cone whoſe baſe is equal to the ſuperficies of the ſector, and its altitude r, which is the laſt of *Lib*. 1.) but we juſt now prov'd that a

ſector is equal to a Cone whoſe altitude is r, and baſe a circle deſcrib'd by the Radius YE, drawn from the vertex of the ſegment EYZ to the circumference of the baſe. Wherefore, a Cone whoſe altitude is, and baſe equal to the ſuperficies of the ſector, is equal to a Cone of the ſame altitude, whoſe baſe is a circle deſcrib'd by the Radius YE.

And ſo the ſuperficies of the ſector EYZ is equal to a circle deſcrib'd by the Radius YE. Which, certainly is the principal Theorem of all thoſe that occur in the Books of *Archimedes*, nor is there found a more excellent one in all Geometry; *viz*, *That the ſuperficies of any ſegment of a ſphere is equal to a circle whoſe Radius is a right line drawn from the vertex of the ſegment to the circumference of the baſes:* And hence, *That the ſuperficies of an Hemiſphere is double of the baſe, or equal to two great circles of the ſphere.*

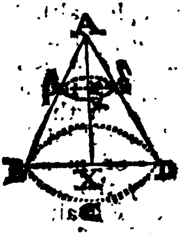
For in this Caſe $\overline{YE}^2 = \overline{AZ}^2 + \overline{AY}^2 = 2\,\overline{AE}^2$ and conſequently a circle deſcribed by the Radius YE is equal to two circles deſcrib'd by the Radius AE. Whence alſo, *the ſuperficies of the whole ſphere is quadruple a circle having the ſame Radius with the ſphere*, that is, *quadruple the greateſt circle in the ſphere; and equal to a circle whoſe Radius is the diameter of the ſphere.* From hence it follows, *That the ſuperficies of a ſphere is equal to the ſuperficies of a Cylinder of the ſame heighth and breadth*; for the ſuperficies of that Cylinder is quadruple to the baſe, as we ſhall ſhew hereafter. And theſe are the moſt noted Theorems of *Archimedes*. Nay, from hence all thoſe things follow, which he



has written concerning the ſuperficies of ſpheres, and their ſegments. So that from theſe few and eaſy Suppoſitions, I have demonſtrated whatever ſeem to be of any Note in the Books *Of the Sphere and Cylinder*.

I will only add, that after by the method of *Archimedes*, (for I think ſcarce any other can be invented, beſides ours, for finding the ſolidity) the ſuperficies of ſegments are found equal to the circle deſcribed by the Radii YE; hence it will plainly follow, that the ſuperficies of ſpheres, and thence of like ſectors are in a duplicate ratio of the Radii of the ſpheres; and conſequently from the ſuperficies thus found, the contents of ſegments, and of whole ſpheres may be mutually deduced, and that very clearly and expeditiouſly after this manner. Becauſe in the ſector EAZ (fig. Pag. 358.) the ſuperficies BZ, CZ, DZ, EZ, proceed as the ſquares deſcrib'd upon AB, AC, AD, AE, that is, as 1, 4, 9, 16, &c.: the whole ſector will be equal to ⅓ of as many ſuperficies equal to the greateſt EZ, or EZ × r, that is, to a Cylinder whoſe baſe is ⅓ EZ, and altitude r, or to a Cone whoſe baſe is EZ, and altitude r. But EZ is ſuppoſed equal to a circle whoſe Radius is YE, wherefore the ſector EAZ is equal to a Cone whoſe baſe is a circle deſcribed by the Radius YZ and altitude r: Which is *Archimedes*'s univerſal Theorem for the contents of *Circles*. Whence if from this the Cone ZAE ſtanding on the baſe of the ſegment EYZ, and having the vertex at the center of the ſphere A, be ſubducted, (you'll have that ſegment EYZ.) But when the ſector EYZ is a Hemiſphere, there will be no ſuch Cone to be ſubducted; and for that reaſon a Cylinder whoſe baſe is ⅓ EZ, and altitude r, or the Cone whoſe baſe is 2 EZ, and Altitude likewiſe ; will be equal to the whole ſphere. But the ſuperficies of the Hemiſphere EZ, is proved to be equal to two of the greateſt circles in the ſphere, whence the whole ſphere is given. This is *Archimedes*'s firſt and principal Theorem, for the content of a ſphere ; whence 'tis eaſily deduced, that a ſphere is $\frac{2}{3}$ of a circumſcrib'd Cylinder, that is, of a Cylinder whoſe altitude and diameter of its baſe is equal to the diameter of the ſphere.

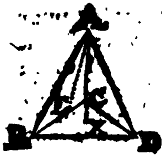
The Doctrine of our other [*Archimedes*] ſeems to make againſt, and ſubvert the new and celebrated *Method of Indiviſibles*, and is preſs'd to that end by *Tacquet*, for inſtance, (*Prop.* 2. *lib.* 2. *Cylindr.*) For the uſual proceſs of that method ſeems to exhibit the dimenſion of the ſuperficies of a Cone, (as alſo of a ſphere, and of other Curves) different enough from what our Author and others have demonſtrated : As for example, let us ſuppoſe ABCD a right Cone, whoſe Axe, is AX, and baſe BCD, and plane $\beta\chi\delta$ drawn, at pleaſure, parallel to the baſe BCD. And ſince, as *Diam.* BD : *Periph.* BCD :: *Diam.* $\beta\delta$: *Periph.* $\beta\chi\delta$; and ſo every where it will be (according to the *Method of Indiviſibles*, and by 12. 5.) as *Diam.* BD, to *Periph.* BCD, ſo is the triangle ABD, conſiſting of thoſe parallel Diameters, to the *Conic Superficies* ABCD, conſiſting of thoſe Peripheries, *i. e. Diam.* BD : *Periph.* BCD :: $\frac{AX \times BD}{2}$: $\frac{AX \times Periph.\ BCD}{2}$ Whence



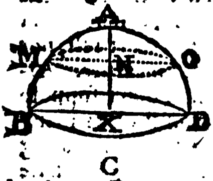
AX

$\frac{AX \times Periph.\ BCD}{2}$ will be equal to the ſuperficies of the Cone; which is falſe and contrary to what was demonſtrated juſt now. For we demonſtrated that the ſuperficies of the Cone was $\frac{AB \times Periph.\ BCD}{2}$.

In anſwering this Objection, we ſay, that the *Method of Indiviſibles*, in the ſpeculation of Perimeters, and of Curve Surfaces, proceeds otherwiſe than in the ſpeculation of plane Surfaces and ſolid Contents. It does indeed ſuppoſe that the Area of plane Figures conſiſt, as it were of parallel right lines, and the contents of ſolids of parallel Planes, and that their number may be expreſs'd by the altitude of the Figures: But it by no means ſuppoſes, that the Perimeters of plane figures conſiſt of points, or the ſuperficies of ſolids, of lines, the number of which may be expreſs'd by the altitude of the figure. As for example, altho' the triangle ABD (in the laſt figure) conſiſts of lines parallel to BD, the number of which is expreſſed by the number of points in the perpendicular AX, that is, by the length of the perpendicular: Yet it would be abſurd to ſuppoſe that the line AB conſiſts of points, whoſe number may be expreſs'd by the number of points in a leſs line AX. For altho' the right line βξ drawn thro' each infinitely ſmall part of AX, divide AB into as many infinitely ſmall parts, yet thoſe parts are not of the ſame Denomination or Quality with the parts of AX, but ſomewhat greater than them; ſo that if the parts of AX be look'd upon as points, the parts of AB are not to be called points, but greater than points; and on the contrary, if the parts of AB be called points, the parts of AX are to be look'd upon as leſs than points, if it be lawful to ſpeak ſo. For the points which are treated of in the *Method of Indiviſibles* are not abſolutely points, but indefinitely ſmall parts, which uſurp the names of points, becauſe of the Affinity. Since therefore points don't admit of greater and leſs, the name of points is not at the ſame time to be attributed to the parts of different magnitudes; conſequently tho' the number of the greater parts of AB may be expreſs'd by the number of the leſſer parts of AX, yet the number of points in AB can no ways be expreſſed by the number of points in AX, (that is, by the number of parts in AX, equal to the number of parts in AB, which are called points.) The line AB has as many points as there are in it ſelf alone, or another line equal to it ſelf, nor can it be determin'd by any other meaſure. After the ſame manner, this method don't ſuppoſe the conic Surface ABCD to conſiſt of as many parallel circumferences perpetually increaſing from the vertex A, or decreaſing from the baſe BD, as there are points in the Axe AX; but rather of as many thus increaſing or decreaſing as there are points in the ſide AB. For in the Revolution of the line AB about the Axis AX, (whereby the ſuperficies of the Cone is generated) every point in the line AB produces a circumference, and conſequently more circumferences are produced than the points contained in the Axis AX. Therefore if you would extend the *Method of Indiviſibles* to the ſuperficies of ſolids, and ſuppoſe thoſe ſuperficies to conſiſt of parallel lines, you ought not to compute this by the parallel Areas conſtituting the ſolid, that is, not to number thoſe Areas by the altitude of the ſolid, but by other lines agreeable to the condition of each figure. Which lines, in figures that are not irregular, may eaſily be determin'd: For inſtance,



ſtance, in the equilateral Pyramid ABCD, whoſe Axe is AX, ſuppoſing that the lateral ſurface of the Pyramid conſiſts of Perimeters of triangles, parallel to the baſe BCD, theſe can neither be computed by the altitude AX, nor by the ſide AB, (for by the former, the thing requir'd, would be wanting of the true Dimenſion, and by the latter 'twould exceed it,) but by the line AE drawn from the vertex A perpendicular to the ſide BC of the baſe; the reaſon of which is, that every plane ſide of a Pyramid as ABC, conſiſts of parallel right lines computed by the altitude AE. After the ſame manner, ſuppoſing that the ſuperficies of the Hemiſphere BAD, conſiſt of peripheries of circles parallel to the baſe BCD, the number of them is not to be computed by the Axis AX, but by the Quadrantal Arc AB, becauſe that every point of the Arc AD in revolving produces a circumference. And ſo any ſuperficies, whether plane or curv'd, which is conceived to conſiſt of equidiſtant right or curv'd lines, is to be computed by a line cutting thoſe equidiſtant lines perpendicularly. For ſince theſe equidiſtant lines, in this *Method of Indiviſibles*, are not conſider'd abſolutely as lines having an infinitely ſmall breadth, which is the ſame with the breadth or thickneſs of the point deſcribing thoſe equidiſtant lines in their Circumvolution, and ſince the ſame equidiſtant lines divide the line cutting them perpendicularly into parts meaſuring its breadth, thoſe parts are to be look'd upon as ſuch ſort of points, and conſequently the number of equidiſtant lines, or the ſum of thoſe breadths is to be computed by the number of points in the line



cutting them perpendicularly, that is, by the length of that line, and not by a line of any other length, for that will conſiſt of more or leſs points.

Hence therefore in the ſpeculation of the ſuperficies of ſolids, the *Method of Indiviſibles* is not unuſeful, but rather very commodious, provided it be rightly underſtood, and applied according to the Rule preſcrib'd. For by the help of it even theſe ſuperficies may be found, if ſo be we have ſome convenient *Data* preſuppos'd, on which the reaſoning may be founded. For inſtance, we might by the help of it, inveſtigate the ſuperficies of a Cone, by reaſoning after this manner.

If the ſuperficies of the cone ABC (fig. pag. 362.) be divided into innumerable Peripheries of circles $\beta\chi\delta$ parallel to the baſe BCD, the breadth of thoſe Peripheries taken together, make up the ſide AB cutting them perpendicularly, and conſequently there will be as many Peripheries as there are points in the line AB, that is, their number may be expreſs'd by the number of points in AB, or by its length. Wherefore, if you draw perpendiculars equal to the Peripheries to every point of AB, a ſuperficies will be made out of thoſe perpendiculars equal to the ſuperficies of the Cone. But that ſuperficies will be a triangle whoſe heighth is AB, and baſe equal to the greateſt Periphery BDC, and ſo the ſuperficies of the Cone will be $= \frac{1}{2}$ AB × *Periph.* BDC, which concluſion agrees with the things laid down and demonſtrated by *Archimedes*.

After the ſame manner, if you take any right line $\alpha\beta$ equal to the quadrantal Arc AB of the Hemiſphere (*in pag.* 364.) and to each of its points μ let the per-